DEVELOPMENTS ASSOCIATED WITH A BOUNDARY PROBLEM NOT LINEAR IN THE PARAMETER*

BY

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I. INTRODUCTION

The expansion of a more or less arbitrary function into a series whose terms are the characteristic functions of a differential equation containing a parameter and subject to certain boundary conditions is a problem which has long claimed the attention of mathematicians.

The oldest of such expansions is the classical Fourier's series, of which the terms satisfy the differential system

$$u''(x)+\lambda u(x)=0,$$
$$u(a)=u(b),$$
$$u'(a)=u'(b),$$

although the system

$$\frac{d}{dx}\left(k\frac{du}{dx}\right)+(\lambda g-l)u=0,$$
$$\alpha_1 u(a)+\alpha_2 u'(a)=0,$$
$$\beta_1 u(b)+\beta_2 u'(b)=0,$$

was used as a source of characteristic functions as early as 1835. Liouville,† using the results of Sturm, was the first to investigate this latter type of development, and with varying hypotheses regarding the coefficients k, g, and l, his work has been carried on by many others, so that with suitable restrictions on these coefficients a fairly complete theory of the expansions has been evolved.

In 1908 Birkhoff‡ broke away from the system of Sturm and Liouville by raising the order of the differential equation from 2 to n and omitting the

* Presented to the Society, February 25, 1922.

† Journal de Mathématiques, vol. 1 (1836).

‡ These Transactions, vol. 9 (1908), p. 373.

condition that it be self adjoint.* The system whose solutions form the terms of his developments may accordingly be written

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + p_{n-1} \frac{du}{dx} + (p_n + \lambda g) u = 0,$$
$$W_1(u) = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$W_n(u) = 0,$$

the W's representing linear homogeneous expressions in u and its first $(n-1)$ derivatives taken at the points a and b.

In several cases the boundary conditions have been changed from the forms here indicated, but only a single further modification of the differential equation appears to have been made. I refer to the equations and conditions

$$p(x)u' + (q + \lambda)u = K(x),$$
$$u(0) = \gamma, \qquad u(1) = \Gamma,$$

and

$$\frac{d}{dx}(pu') + (q + \lambda^2)u = K(x),$$
$$u(0) = \gamma_1, \qquad u'(0) = \gamma_2,$$
$$h_1 u'(1) - h_2 u(1) = \gamma,$$

employed by Hilb in 1911.†

Apparently, therefore, the differential equations hitherto used for this purpose are all reducible to a system of first order equations of the type

$$a_i u_i' = L_i + \lambda M_i \qquad (i = 1, 2, \cdots, n) \tag{1}$$

each L_i and M_i representing a linear expression in $u_1, u_2, \cdots, u_n$.‡ They have in common, moreover, the characteristics

$$a_i \not\equiv 0 \qquad (i = 1, 2, \cdots, n)$$
$$M_i \equiv 0 \text{ if } i \neq n, \qquad M_i \not\equiv 0 \text{ if } i = n.$$

* In this connection see also: Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1904, 1905, 1906; Westfall, *Zur Theorie der Integralgleichungen*, Dissertation, Göttingen, 1905; Bounitsky, *Sur la fonction de Green*, Journal de Mathématiques, 1909.

† Journal für Mathematik, vol. 140, p. 205.

‡ See also Schur, *Zur Entwicklung willkürlicher Funktionen*, Mathematische Annalen, vol. 82[1] (1921), p. 213; Carmichael, *Boundary value and expansion problems*, American Journal of Mathematics, vol. 43 (1921), p. 69.

It is the purpose of this paper on the other hand to consider the expansion problem associated with an equation which when reduced to a system of the form (1) is characterized by the facts that

$$a_i \equiv 0 \text{ if } i \neq n, \qquad a_i \not\equiv 0 \text{ if } i = n,$$
$$M_i \not\equiv 0 \qquad (i = 1, 2, \cdots, n).$$

The equation in question with boundary condition is of the form

$$P(\lambda)\, u'(x) = Q(\lambda, x)\, u(x),$$
$$u(a) = h\, u(b),$$

h being a definite constant (specified below), while P and Q are polynomials in the complex parameter λ, the coefficients of P being constants while those of Q are functions of x which satisfy certain conditions to be specified below. It will furthermore be assumed that the poles of the rational function $Q(\lambda, x)/P(\lambda)$ are all simple.*

II. The formal expansion

Under the assumption already made, there is no further loss of generality in assuming the degree of $P(\lambda)$ to be at least as high as that of $Q(\lambda, x)$.† Accordingly we have the expression in partial fractions

$$\frac{Q(\lambda, x)}{P(\lambda)} = a_0(x) + \sum_{j=1}^{n} \frac{a_j(x)}{\lambda - \alpha_j}.$$

If, now, h is chosen as $h = e^{-\int_a^b a_0(x)\,dx}$, while the dependent variable is changed by the substitution

$$u(x) = e^{\int_a^x a_0(x)\,dx}\, \overline{u}(x),$$

* Observe that the system

$$P(\lambda)\, \overline{u}'(x) = Q(\lambda, x)\, \overline{u}(x),$$
$$\overline{u}(a) = h_1 e^{-\frac{q(\lambda)}{p(\lambda)}}\, \overline{u}(b),$$

where q and p are polynomials with constant coefficients, while $q(\lambda)/p(\lambda)$ has only simple poles, is not more general than system (1) since it may be reduced to (1) by the substitution $u = \overline{u}\, e^{\frac{q(\lambda)x}{(a-b)p(\lambda)}}$.

† The degree of Q can exceed that of P by at most one, while if Q is of higher degree than P the change of parameter $\lambda = \frac{1}{\bar{\lambda}} + \beta$, where $(\lambda - \beta)$ is not a factor of $P(\lambda)$, may be made and will raise the degree of P to that of Q.

the system takes the form

$$u'(x) = \sum_{j=1}^{n} \frac{a_j(x)}{\lambda - \alpha_j} u(x), \qquad u(a) = u(b). \tag{2}$$

This is the form which will be employed hereafter, and it will be assumed that the coefficients of $Q(\lambda, x)$ are such that each of the functions $a_j(x)$ $(j = 1, 2, \cdots, n)$ in (2) is real, maintains the same sign, and is integrable and bounded throughout a closed interval $a \leq x \leq b$.

As a matter of notation the capital will be used to indicate the integral from a to x of the corresponding small letter; thus

$$\Omega(x) = \int_a^x \omega(x)\, dx.$$

Accordingly the solution of equation (2) may be written

$$u(x) = e^{\sum_{j=1}^{n} \frac{A_j(x)}{\lambda - \alpha_j}},*$$

and the characteristic values of λ for which this solution satisfies the boundary conditions are found to be the roots of the equations

$$\sum_{j=1}^{n} \frac{A_j(b)}{\lambda - \alpha_j} = 2k\pi i, \tag{3}$$

k being any integer or zero.

It will be assumed throughout the discussion unless the contrary is explicitly stated that equations (3) have no multiple roots, a condition which is always fulfilled when none of the roots of the equation

$$\sum_{j=1}^{n} \frac{A_j(b)}{(\lambda - \alpha_j)^2} = 0$$

are characteristic values.†

Equations (3) yield, therefore, n characteristic values $\lambda_{k_1}, \lambda_{k_2}, \cdots, \lambda_{k_n}$, corresponding to each choice of k. We shall return subsequently to a discussion of these values and will show that they cluster about the points $\alpha_1, \alpha_2, \cdots, \alpha_n$,

* The arbitrary constant factor will be omitted.

† In case two or more characteristic values do coincide the corresponding terms of the expansion must be otherwise computed (see note on p. 15).

for $|k|$ large, a sufficiently small neighborhood of α_ν containing just one characteristic value λ_{k_ν}, for each integer k.

Permitting λ to take on characteristic values in the equations of system (2) and of the adjoint system

$$(4)\qquad \begin{aligned} v'(x) &= -\sum_{j=1}^{n} \frac{a_j(x)}{\lambda - \alpha_j} v(x),^* \\ v(a) &= v(b), \end{aligned}$$

we have

$$u'_{l_\mu}(x) = \sum_{j=1}^{n} \frac{a_j(x)}{\lambda_{l_\mu} - \alpha_j} u_{l_\mu}(x),$$

$$v'_{k_\nu}(x) = -\sum_{j=1}^{n} \frac{a_j(x)}{\lambda_{k_\nu} - \alpha_j} v_{k_\nu}(x).$$

If, now, these equations are multiplied by $v_{k_\nu}(x)$ and $u_{l_\mu}(x)$ respectively and are then added, it is found that

$$\begin{aligned} \frac{d}{dx}(u_{l_\mu} v_{k_\nu}) &= \sum_{j=1}^{n} a_j(x) \left[\frac{1}{\lambda_{l_\mu} - \alpha_j} - \frac{1}{\lambda_{k_\nu} - \alpha_j} \right] u_{l_\mu} v_{k_\nu} \\ &= (\lambda_{k_\nu} - \lambda_{l_\mu}) \sum_{j=1}^{n} \frac{a_j(x)}{(\lambda_{l_\mu} - \alpha_j)(\lambda_{k_\nu} - \alpha_j)} u_{l_\mu} v_{k_\nu}. \end{aligned}$$

The quantity $u_{l_\mu} v_{k_\nu}$, however, takes on identical values at the boundary points a and b, whence integration between these limits yields the fact that

$$(5)\qquad \int_a^b \sum_{j=1}^{n} \frac{a_j(x)}{(\lambda_{l_\mu} - \alpha_j)(\lambda_{k_\nu} - \alpha_j)} u_{l_\mu} v_{k_\nu} dx = 0,$$

unless $l = k$ and $\mu = \nu$. We proceed to apply the relation thus established.

Let $f_1(x), f_2(x), \cdots, f_n(x)$, be n arbitrary functions, and let us assume the possibility of expanding these functions simultaneously in series of the form

$$(6)\qquad f_j(x) = \sum_{\substack{l=-\infty \\ \mu=1,2,\ldots,n}}^{+\infty} c_{l_\mu} \frac{u_{l_\mu}(x)}{\lambda_{l_\mu} - \alpha_j} \qquad (j = 1, 2, \cdots, n).$$

* The characteristic values of systems (2) and (4) are identical.

Multiplying by $a_j(x)v_{k_\nu}(x)/(\lambda_{k_\nu}-\alpha_j)$ and summing with respect to j we obtain the relation

$$\sum_{j=1}^{n}\frac{a_j(x)f_j(x)\,v_{k_\nu}(x)}{\lambda_{k_\nu}-\alpha_j}=\sum_{j=1}^{n}\sum_{\substack{l=-\infty\\ \mu=1,2,\ldots,n}}^{+\infty}c_{l_\mu}\frac{a_j(x)\,u_{l_\mu}(x)\,v_{k_\nu}(x)}{(\lambda_{k_\nu}-\alpha_j)\,(\lambda_{l_\mu}-\alpha_j)}.$$

On the assumption that the series on the right of this equation converge and may be integrated term by term between the points a and b it is found in view of relation (5) and the fact that $u_{k_\nu}(x)\,v_{k_\nu}(x)\equiv 1$, that

$$\int_a^b\sum_{j=1}^{n}\frac{a_j(x)f_j(x)\,v_{k_\nu}(x)}{(\lambda_{k_\nu}-\alpha_j)}\,dx=c_{k_\nu}\sum_{j=1}^{n}\frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}.$$

Inasmuch as the sum on the right was assumed to be different from zero,

$$c_{k_\nu}=\frac{\displaystyle\int_a^b\sum_{j=1}^{n}\frac{a_j(x)f_j(x)\,v_{k_\nu}(x)}{\lambda_{k_\nu}-\alpha_j}\,dx}{\displaystyle\sum_{j=1}^{n}\frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}},\tag{7}$$

and (6) is more explicitly

$$f_i(x)=\sum_{\substack{k=-\infty\\ \nu=1,2,\ldots,n}}^{+\infty}\int_a^b\sum_{j=1}^{n}\frac{a_j(t)f_j(t)\,v_{k_\nu}(t)\,u_{k_\nu}(x)}{(\lambda_{k_\nu}-\alpha_j)\,(\lambda_{k_\nu}-\alpha_i)\displaystyle\sum_{j=1}^{n}\frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}}\,dt.\tag{8}$$

Instead of returning to a rigorous establishment of this formal result we shall investigate directly the questions, does the series in (8) converge? and if so, what does it represent? Before proceeding to this, however, it is of interest to observe the nature of the reduction of expansion (8) in a particularly simple case, namely that in which the coefficients $a_j(x)$ are all constants. For simplicity let n be chosen as $n=2$.

From (3) the characteristic values λ_{k_1} and λ_{k_2} are the roots of the equation

$$\frac{a_1}{\lambda-\alpha_1}+\frac{a_2}{\lambda-\alpha_2}=\frac{2k\pi i}{(b-a)}$$

and since, therefore,

$$\frac{a_1}{\lambda_{k_1}-\alpha_1}+\frac{a_2}{\lambda_{k_1}-\alpha_2}=\frac{a_1}{\lambda_{k_2}-\alpha_1}+\frac{a_2}{\lambda_{k_2}-\alpha_2},$$

it is readily seen that

$$a_2\left(\frac{\lambda_{k_1}-\alpha_1}{\lambda_{k_1}-\alpha_2}\right) = -a_1\left(\frac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}\right). \tag{9}$$

Furthermore, since a_1 and a_2 are constants,

$$u_{k_1}(x) \equiv u_{k_2}(x) \equiv e^{2k\pi i\frac{x-a}{b-a}}.$$

The expansion (8) for $f_1(x)$ is, therefore,

$$f_1(x) = \sum_{k=-\infty}^{+\infty}\int_a^b \left\{\frac{\dfrac{a_1 f_1(t)}{(\lambda_{k_1}-\alpha_1)^2}+\dfrac{a_2 f_2(t)}{(\lambda_{k_1}-\alpha_2)(\lambda_{k_1}-\alpha_1)}}{\left[\dfrac{a_1}{(\lambda_{k_1}-\alpha_1)^2}+\dfrac{a_2}{(\lambda_{k_1}-\alpha_2)^2}\right](b-a)}\right.$$

$$\left. + \frac{\dfrac{a_1 f_1(t)}{(\lambda_{k_2}-\alpha_1)^2}+\dfrac{a_2 f_2(t)}{(\lambda_{k_2}-\alpha_2)(\lambda_{k_2}-\alpha_1)}}{\left[\dfrac{a_1}{(\lambda_{k_2}-\alpha_1)^2}+\dfrac{a_2}{(\lambda_{k_2}-\alpha_2)^2}\right](b-a)}\right\} v_k(t)\,u_k(x)\,dt.$$

Multiplying both numerator and denominator of each fraction of the integrand by a suitable factor, the expression within the brace takes the form

$$\left\{\frac{f_1(t)+f_2(t)\dfrac{a_2}{a_1}\left(\dfrac{\lambda_{k_1}-\alpha_1}{\lambda_{k_1}-\alpha_2}\right)}{1+\dfrac{a_2}{a_1}\left(\dfrac{\lambda_{k_1}-\alpha_1}{\lambda_{k_1}-\alpha_2}\right)^2}+\frac{f_1(t)\dfrac{a_1}{a_2}\left(\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}\right)^2+f_2(t)\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}}{\dfrac{a_1}{a_2}\left(\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}\right)^2+1}\right\}.$$

This reduces on account of (9) to

$$\left\{\frac{f_1(t)-f_2(t)\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}+f_1(t)\dfrac{a_1}{a_2}\left(\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_1}-\alpha_1}\right)^2+f_2(t)\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}}{1+\dfrac{a_1}{a_2}\left(\dfrac{\lambda_{k_2}-\alpha_2}{\lambda_{k_2}-\alpha_1}\right)^2}\right\}$$

namely to $f_1(t)$, and the expansion in question may be written

$$f_1(x) = \sum_{k=-\infty}^{+\infty}\frac{1}{b-a}\int_a^b f_1(t)\,e^{2k\pi i\frac{x-t}{b-a}}\,dt.$$

Inasmuch as the imaginary terms of the series for the values of k numerically equal but of opposite sign cancel, if $f_1(x)$ is real, the series for such a function reduces to

$$f_1(x) = \sum_{k=-\infty}^{+\infty} \frac{1}{b-a} \int_a^b f_1(t) \cos \frac{2k\pi}{b-a}(x-t)\,dt.$$

Hence expansion (8) for $f_1(x)$ is none other than its Fourier's series, and in precisely similar manner the reduction of (8) to the Fourier's series in the case of $f_2(x)$ may be shown. The generalization is, therefore, one from constant coefficients a_j to variable coefficients $a_j(x)$.

III. The expansion as a contour integral

When λ is not a characteristic value of system (2) there exists a unique Green's function, $G(x, t, \lambda)$, which is characterized by the fact that the solution of the non-homogeneous system

$$y'(x) = \sum_{j=1}^{n} \frac{a_j(x)}{\lambda - a_j} y(x) + \omega(x),$$

$$y(a) = y(b),$$

is given by the formula

$$y(x) = \int_a^b G(x, t, \lambda)\, \omega(t)\, dt.^*$$

The explicit formula for $G(x, t, \lambda)$ is in this case

$$G(x, t, \lambda) = \frac{u(x, \lambda)\, v(t, \lambda)}{2} \left[\pm 1 + \frac{1 + u(b, \lambda)}{1 - u(b, \lambda)} \right], \tag{10}$$

where the ambiguous sign must be chosen

$$+ \text{ for } t < x, \qquad - \text{ for } t > x,$$

and where

$$u(x, \lambda) = e^{\sum_{j=1}^{n} \frac{A_j(x)}{\lambda - a_j}},$$

while

$$u(x, \lambda)\, v(x, \lambda) \equiv 1.$$

* Bôcher, Bulletin of the American Mathematical Society. vol. 7 (1901), p. 297.

It is seen, therefore, that $G(x, t, \lambda)$ considered as a function of the parameter λ is analytic over the entire plane except for a pole at each of the characteristic values, these poles being of the first order when the characteristic equations have only simple roots. The residue at $\lambda = \lambda_{k_\nu}$ is, of course, given by the formula

$$R_{k_\nu}(x,t) = \frac{u(x,\lambda_{k_\nu})v(t,\lambda_{k_\nu})}{2} \frac{1+u(b,\lambda_{k_\nu})}{\frac{d}{d\lambda}[1-u(b,\lambda)]_{\lambda=\lambda_{k_\nu}}},$$

or, since

$$u(x,\lambda_{k_\nu}) = u_{k_\nu}(x),$$

more explicitly by

$$R_{k_\nu}(x,t) = \frac{u_{k_\nu}(x)\, v_{k_\nu}(t)}{\sum_{j=1}^{n} \frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}}.$$

Hence if γ_{k_ν} is any closed contour in the λ plane which includes λ_{k_ν} and no other characteristic values, and none of points α_i $(i = 1, 2, \cdots, n)$, then

$$\frac{1}{2\pi i}\int_{\gamma_{k_\nu}} \frac{G(x,t,\lambda)}{(\lambda-\alpha_j)(\lambda-\alpha_i)}\, d\lambda = \frac{v_{k_\nu}(t)\, u_{k_\nu}(x)}{(\lambda_{k_\nu}-\alpha_j)(\lambda_{k_\nu}-\alpha_i)\sum_{j=1}^{n}\frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}}.$$

Multiplying this by $a_j(t)f_j(t)$, summing with respect to j, and integrating with respect to t, we obtain the relation

$$\frac{1}{2\pi i}\int_{\gamma_{k_\nu}}\int_a^b \sum_{j=1}^{n} \frac{a_j(t)f_j(t)\,G(x,t,\lambda)}{(\lambda-\alpha_j)(\lambda-\alpha_i)}\, dt\, d\lambda = \int_a^b \sum_{j=1}^{n} \frac{a_j(t)f_j(t)\,v_{k_\nu}(t)\,u_{k_\nu}(x)}{(\lambda_{k_\nu}-\alpha_j)(\lambda_{k_\nu}-\alpha_i)\sum_{j=1}^{n}\frac{A_j(b)}{(\lambda_{k_\nu}-\alpha_j)^2}}\, dt.$$

The right-hand side of this equation is, however, precisely the k_νth term of the series in (8)* from which it follows that if γ_m is a contour enclosing the

* If a number of characteristic values coincide and are located at λ_{k_ν} the corresponding terms of series (8) taken together are to be replaced by

$$\int_a^b \sum_{j=1}^{n} \frac{a_j(t)f_j(t)R_{k_\nu}(x,t)}{(\lambda_{k_\nu}-a_j)(\lambda_{k_\nu}-a_i)}\, dt,$$

where $R_{k_\nu}(x, t)$ is the residue of $G(x, t, \lambda)$ at $\lambda = \lambda_{k_\nu}$.

characteristic values $\lambda_1, \lambda_2, \cdots, \lambda_m$, and no others, then $S_m^{(i)}(x)$, the sum of the terms of series (8) corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_m$ is given by

$$S_m^{(i)}(x) = \frac{1}{2\pi i} \int_{\gamma_m} \int_a^b \sum_{j=1}^n \frac{a_j(t) f_j(t) G(x,t,\lambda)}{(\lambda - \alpha_j)(\lambda - \alpha_i)} \, dt \, d\lambda. \tag{11}$$

IV. Convergence of the series

The equation for the characteristic values has already been given, (3), and it was observed that these values cluster about the n points $\alpha_1, \alpha_2, \cdots, \alpha_n$, for $|k|$ large. The substitution

$$\frac{1}{\lambda - \alpha_\tau} = \varrho \tag{12}$$

projects the point α_τ to ∞ and transforms (3) into

$$\sum_{j=1}^n \frac{A_j(b)}{\alpha_\tau - \alpha_j + 1/\varrho} = 2k\pi i, \tag{13}$$

which may also be written

$$\varrho_{k_\tau} = \frac{1}{A_\tau(b)} \left\{ 2k\pi i + \sum_{\substack{j=1 \\ j \neq \tau}}^n \left[\frac{A_j(b)}{\alpha_j - \alpha_\tau} + 1/\varrho \frac{A_j(b)}{(\alpha_\tau - \alpha_j)(\alpha_\tau - \alpha_j + 1/\varrho)} \right] \right\}.$$

From this every characteristic value is seen to be of the form

$$\varrho_{k_\tau} = \frac{2k\pi i + C}{A_\tau(b)} + \frac{\sigma(\varrho)}{\varrho}, \tag{14}$$

C being a constant and $\sigma(\varrho)$ being bounded for $|\varrho| > R$.

Consider a small circle of fixed radius ϵ drawn about the points $\varrho = \dfrac{2k\pi i + C}{A_\tau(b)}$, where the quantity C is that of formula (14). For $|\varrho|$ sufficiently large the point

$$\frac{2k\pi i + C}{A_\tau(b)} + \frac{\sigma(\varrho)}{\varrho}$$

will lie within this circle for any ϱ on the circumference, and hence, as ϱ describes this circumference.

$$\text{arc} \left\{ \varrho - \left(\frac{2k\pi i + C}{A_\tau(b)} + \frac{\sigma(\varrho)}{\varrho} \right) \right\}$$

increases by 2π. It follows, therefore, that for $|\varrho| > R$ the characteristic values all lie within these small circles, and that each circle contains just one characteristic value.

It is clear from this that the interval between any two characteristic values given by (14) for consecutive values of k becomes more and more nearly equal as $|k|$ is taken larger and larger, and that it is possible, therefore, to choose a set of circles $\bar{C}_j$ with centers in $\varrho = 0$ and of successively larger radii (there being always one whose radius exceeds any given constant) which are such that the distance between any point of a circle $\bar{C}_j$ and any characteristic value exceeds some constant $\delta > 0$. Let C_j be used now to denote any circle of the set into which the $\bar{C}_j$'s are transformed by the inverse of substitution (12). Having defined the set C_j in this manner let the procedure be repeated for each of the remaining points $\alpha_1, \cdots, \alpha_n$, and the n sets of circles $C_1, C_2, \cdots, C_n$ be thus determined. It can easily be shown that there exists a positive lower limit M for the values of

$$|1 - u(b, \lambda)|, \text{ and } |v(b, \lambda) - 1|,$$

when λ lies on any of the circles C_i $(i = 1, 2, \cdots, n)$.

Returning now to formula (11), let the contour γ_m be composed of circles C chosen in the manner described above, one such circle being drawn about each of the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and described in the negative sense. Then $S_m^{(i)}(x)$, as given by the formula

$$(15) \qquad S_m^{(i)}(x) = \frac{1}{2\pi i} \int_{-C_1 - C_2 - \cdots - C_n} \int_a^b \sum_{j=1}^{n} \frac{a_j(t) f_j(t) G(x, t, \lambda)}{(\lambda - \alpha_j)(\lambda - \alpha_i)} \, dt \, d\lambda,$$

will represent the sum of the terms corresponding to the characteristic values excluded by all the circles in question. As the circles are taken smaller and smaller, (15) will sum a larger and larger number of terms, and the limit of the integral as the process of successively replacing the circles C by smaller circles C is allowed to continue indefinitely will, if it exists, represent the sum of the series. We proceed to the evaluation of this limit.

Upon writing (15) in the form

$$(15a) \qquad S_m^{(i)}(x) = \sum_{l=1}^{n} \frac{1}{2\pi i} \int_{-C_l} \int_a^b \frac{G(x, t, \lambda)}{\lambda - \alpha_i} \sum_{j=1}^{n} \frac{a_j(t) f_j(t)}{\lambda - \alpha_j} \, dt \, d\lambda,$$

we have in the sum on the right integrals of two types, namely those in which j and l are not both equal to i, and that for which j, l, and i are all

equal. In the evaluation of these integrals it will be convenient to refer to the fact established by

LEMMA I. *Given* $\varphi(t,\varrho)$ *any function such that* (i) $|\varphi(t,\varrho)|<K$ *for* $\alpha\leqq t\leqq\beta$, $\gamma\leqq\theta\leqq\delta$, $|\varrho|>R$, *and* (ii) $\lim_{|\rho|\to\infty}|\varphi(t,\varrho)|=0$ *uniformly for* $\alpha+\varepsilon\leqq t\leqq\beta-\varepsilon$, $\gamma+\varepsilon\leqq\theta\leqq\delta-\varepsilon$, ε *being any sufficiently small positive constant, and* θ *denoting arc* ϱ. *Then if*

$$I=\frac{1}{2\pi i}\int_C\int_\alpha^\beta\varphi(t,\varrho)\,dt\,\frac{d\varrho}{\varrho},$$

C being that arc of circle $|\varrho|=r$ *drawn from the point* $\theta=\gamma$ *to the point* $\theta=\delta$, *it follows that*

$$\lim_{\rho\to\infty}I=0.$$

PROOF. Setting $\varrho=re^{i\theta}$, and defining the function ψ by the relation $\varphi(t,\varrho)=\psi(t,\theta,r)$ we have

$$I=\frac{1}{2\pi}\int_\gamma^\delta\int_\alpha^\beta\psi(t,\theta,r)\,dt\,d\theta;$$

whence

$$|I|\leqq\frac{1}{2\pi}\left\{\left|\int_{\gamma+\varepsilon}^{\delta-\varepsilon}\int_\alpha^{\beta-\varepsilon}\psi(t,\theta,r)\,dt\,d\theta\right|+\int_\alpha^\beta\int_\gamma^{\gamma+\varepsilon}K\,d\theta\,dt+\int_\alpha^\beta\int_{\delta-\varepsilon}^{\delta}K\,d\theta\,dt\right.$$
$$\left.+\int_{\gamma+\varepsilon}^{\delta-\varepsilon}\int_\alpha^{\alpha+\varepsilon}K\,dt\,d\theta+\int_{\gamma+\varepsilon}^{\delta-\varepsilon}\int_{\beta-\varepsilon}^{\beta}K\,dt\,d\theta\right\}.$$

From this it is apparent, however, inasmuch as the limit of the first integral on the right is zero, that for r sufficiently large

$$|I|<\frac{2lK}{\pi}\varepsilon,\qquad\text{where }\begin{cases}l\geqq|\beta-\alpha|,\\ l\geqq|\delta-\gamma|.\end{cases}$$

Since ε was arbitrary, however, this implies that

$$\lim_{\rho\to\infty}I=0.\qquad\text{Q. E. D.}$$

Let us consider now the integrals in formula (15a), assuming the functions $f_j(x)$ which we wish to develop to consist, in the interval $a\leqq x\leqq b$, of at

most a finite number of pieces, each continuous and having a continuous first derivative. We have

LEMMA II. *If*

$$I_2 = \frac{1}{2\pi i} \int_{-C_\tau}^{} \int_{a}^{b} \frac{G(x,t,\lambda)\, a_\mu(t) f_\mu(t)}{(\lambda - \alpha_\nu)(\lambda - \alpha_\mu)}\, dt\, d\lambda,$$

where

$$(\tau, \mu) \neq (\nu, \nu),$$

then

$$\lim I_2 = 0,$$

when C_τ is taken successively as a smaller and smaller circle of the type C defined above, drawn about the point α_τ.

PROOF. Transforming the plane of the parameter by the substitution (12) we have

$$I_2 = \frac{-1}{2\pi i} \int_{\bar{C}_\tau}^{} \int_{a}^{b} \Phi(t,\varrho)\, G(x,t,\alpha_\tau + 1/\varrho)\, dt \frac{d\varrho}{\varrho}, \tag{16}$$

where $\Phi(t,\varrho)$, being given by the formula

$$\Phi(t,\varrho) = \frac{a_\mu(t) f_\mu(t)}{\varrho(\alpha_\tau - \alpha_\nu + 1/\varrho)(\alpha_\tau - \alpha_\mu + 1/\varrho)},$$

is seen to be bounded for $|\varrho| > R$, since $a_\mu(t)$ and $f_\mu(t)$ are bounded, while at least one of the quantities α_ν and α_μ differs from α_τ. $\bar{C}_\tau$ is now a circle $|\varrho| = r$. The expression for the Green's function, $G(x,t,\alpha_\tau + 1/\varrho)$, is given in (10) but may be more conveniently used in the following equivalent forms:

$$G(x,t,\alpha_\tau + 1/\varrho) = \frac{u(x,\varrho)\, v(t,\varrho)}{1 - u(b,\varrho)} = \frac{u(x,\varrho)\, v(t,\varrho)\, v(b,\varrho)}{v(b,\varrho) - 1}$$

when $t < x$, and

$$G(x,t,\alpha_\tau + 1/\varrho) = \frac{u(x,\varrho)\, v(t,\varrho)\, u(b,\varrho)}{1 - u(b,\varrho)} = \frac{u(x,\varrho)\, v(t,\varrho)}{v(b,\varrho) - 1}$$

when $t > x$.

Denoting by $\bar{C}'_\tau$ the arc of $\bar{C}_\tau$ which lies in the half-plane $R\{\varrho\, a_\tau(t)\} \geq 0$,* and by $\bar{C}''_\tau$ the arc on which $R\{\varrho\, a_\tau(t)\} < 0$, I_2 may be more explicitly written as the sum of the integrals

* The notation $R\{\omega\}$ is used to indicate "the real part of ω".

$$I_{21} = \frac{-1}{2\pi i}\int_{\overline{C}_\tau'}\int_a^x \Phi(t,\varrho)\frac{u(x,\varrho)\,v(t,\varrho)\,v(b,\varrho)}{v(b,\varrho)-1}\,dt\,\frac{d\varrho}{\varrho},$$

$$I_{22} = \frac{-1}{2\pi i}\int_{\overline{C}_\tau''}\int_a^x \Phi(t,\varrho)\frac{u(x,\varrho)\,v(t,\varrho)}{1-u(b,\varrho)}\,dt\,\frac{d\varrho}{\varrho},$$

$$I_{23} = \frac{-1}{2\pi i}\int_{\overline{C}_\tau'}\int_x^b \Phi(t,\varrho)\frac{u(x,\varrho)\,v(t,\varrho)}{v(b,\varrho)-1}\,dt\,\frac{d\varrho}{\varrho},$$

$$I_{24} = \frac{-1}{2\pi i}\int_{\overline{C}_\tau''}\int_x^b \Phi(t,\varrho)\frac{u(x,\varrho)\,v(t,\varrho)\,u(b,\varrho)}{1-u(b,\varrho)}\,dt\,\frac{d\varrho}{\varrho}.$$

The notation $u(x,\varrho)$ has been used here in preference to the more clumsy $u(x, a_\tau + 1/\varrho)$.

Let us consider the integrand of I_{21} which is typical. For ϱ on $\overline{C}_\tau'$ the factor $\dfrac{\Phi(t,\varrho)}{v(b,\varrho)-1}$ is bounded, while the remaining factor is

$$u(x,\varrho)\,v(t,\varrho)\,v(b,\varrho) = e^{\rho\left\{A_\tau(x)-A_\tau(t)-A_\tau(b)\right\}+\sum_{\substack{j=1\\ j\neq\tau}}^{n}\frac{A_j(x)-A_j(t)-A_j(b)}{a_\tau-a_j+1/\rho}}.$$

If $x \neq b$ this clearly approaches the limit zero uniformly as $|\varrho|$ increases, for all values of t in the interval $a \leqq t \leqq x$ and ϱ in any closed sector within and on the boundary of which $R\{\varrho\, a_\tau(t)\} > 0$. The same is true also if $a \leqq x \leqq b$ and $a + \epsilon \leqq t \leqq x$, while the value of the expression for $a \leqq x \leqq b$, and all values of t in the interval $a \leqq t \leqq x$ and $R\{\varrho\, a_\tau(t)\} \geqq 0$ is bounded. The entire integrand, is, therefore, for any given x, of the type $\varphi(t,\varrho)$ of Lemma I, and accordingly

$$\lim_{|\rho|\to\infty} I_{21} = 0.$$

Inasmuch as the same reasoning may be applied to each of the remaining integrals I_{2i}, it is readily found that

$$\lim_{|\rho|\to\infty} I_2 = 0. \qquad \text{Q. E. D.}$$

The integral in (15a) which results from $i = l = j$ remains to be considered. We have

LEMMA III. *If*

$$I_3 = \frac{1}{2\pi i}\int_{-C_\tau}\int_a^b \frac{G(x,t,\lambda)\,a_\tau(t)f_\tau(t)}{(\lambda-\alpha_\tau)^2}\,dt\,d\lambda,$$

then

$$\lim I_3 = \begin{cases} \frac{1}{2}\{f_\tau(x+0)+f_\tau(x-0)\} & \text{for } x \neq a,\ x \neq b, \\ \frac{1}{2}\{f_\tau(a+0)+f_\tau(b-0)\} & \text{for } x = a \text{ or } x = b. \end{cases}$$

PROOF. Applying again the transformation (12) we obtain the form

$$I_3 = \frac{-1}{2\pi i}\int_{C_\tau}\int_a^b G(x,t,\alpha_\tau+1/\varrho)\,a_\tau(t)f_\tau(t)\,d\varrho,$$

or, substituting the values of $G(x, t, \alpha_\tau+1/\varrho)$, as given by formula (10),

$$I_3 = \frac{-1}{2\pi i}\int_{C_\tau}\frac{u(x,\varrho)}{1-u(b,\varrho)}\left\{\int_a^x v(t,\varrho)a_\tau(t)f_\tau(t)\,dt + u(b,\varrho)\int_x^b v(t,\varrho)\,a_\tau(t)f_\tau(t)\,dt\right\}d\varrho.$$

The integration by parts indicated by the relation

$$-\int_\gamma^\delta a_\tau(t)f_\tau(t)\,v(t,\varrho)\,dt = 1/\varrho\left\{f_\tau(\delta-0)v(\delta,\varrho) - f_\tau(\gamma+0)v(\gamma,\varrho) - \int_\gamma^\delta\left\{f_\tau'(t) - f_\tau(t)\sum_{\substack{j=1\\ j\neq\tau}}^{n}\frac{a_j(t)}{\alpha_\tau-\alpha_j+1/\varrho}\right\}v(t,\varrho)\,dt\right\}$$

now yields a form for I_3, of which the various terms may be collected into the following expressions:

$$I_{31} = \frac{-1}{2\pi i}\int_{C_\tau}\int_a^b\left\{f_\tau'(t) - f_\tau(t)\sum_{\substack{j=1\\ j\neq\tau}}^{n}\frac{a_j(t)}{\alpha_\tau-\alpha_j+1/\varrho}\right\}G(x,t,\alpha_\tau+1/\varrho)\,dt\,\frac{d\varrho}{\varrho},$$

$$I_{32}=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{\bar{C}_\tau}\Big[(f_\tau(x-0)-[f_\tau(a+0)-f_\tau(b-0)]\,u(x,\varrho)\\ \qquad\qquad -f_\tau(x+0)\,u(b,\varrho))\dfrac{1}{1-u(b,\varrho)}\Big]\dfrac{d\varrho}{\varrho},\\ \text{when } x\neq a,\ x\neq b,\\ \dfrac{1}{2\pi i}\displaystyle\int_{\bar{C}_\tau}\dfrac{f_\tau(b-0)-f_\tau(a+0)\,u(b,\varrho)}{1-u(b,\varrho)}\,\dfrac{d\varrho}{\varrho},\\ \text{when } x=a \text{ or } x=b.\end{cases}$$

Now inasmuch as

$$\Big\{f_\tau'(t)-f_\tau(t)\sum_{\substack{j=1\\ j\neq\tau}}^{n}\frac{a_j(t)}{\alpha_\tau-\alpha_j+1/\varrho}\Big\}$$

is bounded for $|\varrho|>R$, the integral I_{31} is seen to be of precisely the same type as the integral I_2 in formula (16), and it follows accordingly that

$$\lim I_{31}=0.$$

Further, from the more explicit forms

$$I_{32}=\frac{1}{2\pi i}\int_{\bar{C}_\tau'}\Big\{\frac{[f_\tau(x-0)-f_\tau(x+0)]-[f_\tau(a+0)-f_\tau(b-0)]\,u(x,\varrho)}{1-u(b,\varrho)}+f_\tau(x+0)\Big\}\frac{d\varrho}{\varrho}+\frac{1}{2\pi i}\int_{\bar{C}_\tau''}\Big\{f_\tau(x-0)+\frac{[f_\tau(x-0)-f_\tau(x+0)]u(b,\varrho)-[f_\tau(a+0)-f_\tau(b-0)]u(x,\varrho)}{1-u(b,\varrho)}\Big\}\frac{d\varrho}{\varrho}$$

when $x\neq a$, $x\neq b$,

$$I_{32}=\frac{1}{2\pi i}\int_{\bar{C}_\tau'}\Big\{\frac{[f_\tau(b-0)-f_\tau(a+0)]}{1-u(b,\varrho)}+f_\tau(a+0)\Big\}\frac{d\varrho}{\varrho}+\frac{1}{2\pi i}\int_{\bar{C}_\tau''}\Big\{f_\tau(b-0)+\frac{[f_\tau(b-0)-f_\tau(a+0)]u(b,\varrho)}{1-u(b,\varrho)}\Big\}\frac{d\varrho}{\varrho}$$

when $x = a$ or $x = b$, the evaluation of I_{32} is simple. The fractional parts of these integrands are functions of such type that their integrals over the respective paths of integration vanish when the limit as $|\rho| = \infty$ is taken, as may be determined by an argument analogous to that used in the proof of Lemma I. It follows, therefore, that

$$\lim_{|\rho|\to\infty} I_{32} = \tfrac{1}{2}\{f_\tau(x+0)+f_\tau(x-0)\}$$

when $x \neq a$, $x \neq b$,

$$\lim_{|\rho|\to\infty} I_{32} = \tfrac{1}{2}\{f_\tau(a+0)+f_\tau(b-0)\},$$

when $x = a$ or $x = b$. This establishes the final lemma.

THEOREM. *Given any n functions $f_1(x), f_2(x), \cdots, f_n(x)$, which, in the interval $a \leqq x \leqq b$, consist of at most a finite number of pieces, each real and continuous and having a continuous derivative; then these functions may be simultaneously expanded in series of the form*

$$f_i(x) = \sum_{\substack{k=-\infty \\ j=1,2,\cdots,n.}}^{+\infty} C_{kj}\frac{u_{k_j}(x)}{\lambda_{k_j}-\alpha_i} \qquad (i=1,2,\cdots,n).$$

In this the λ's and u's are respectively the characteristic values and functions of a differential system

$$u'(x) = \sum_{j=1}^{n}\frac{a_j(x)}{\lambda-\alpha_j}u(x),$$

$$u(a) = u(b),$$

each α_i being a constant, while each $a_j(x)$ is real, maintains the same sign, and is bounded and integrable throughout the interval $a \leqq x \leqq b$. This expansion for $f_i(x)$ will, moreover, converge to $\frac{1}{2}\{f_i(x+0)+f_i(x-0)\}$ for any interior point of the interval, and to $\frac{1}{2}\{f_i(a+0)+f_i(b-0)\}$ at the end points.

PROOF. From the choice of notation $f_i(x) = \lim_{m\to\infty} S_m^{(i)}(x)$, and hence by (15$a$)

$$f_i(x) = \lim \sum_{j,\tau=1}^{n}\frac{1}{2\pi i}\int_{-C_\tau}\int_a^b \frac{a_j(t)f_j(t)G(x,t,\lambda)}{(\lambda-\alpha_j)(\lambda-\alpha_i)}dt\,d\lambda.$$

The evaluation of this expression may be simplified by writing it in the form

$$f_i(x) = \lim \sum_{\substack{j,\tau=1 \\ (j,\tau)\neq(i,0)}}^{n} \frac{1}{2\pi i} \int_{-C_\tau} \int_a^b \frac{a_j(t) f_j(t) G(x,t,\lambda)}{(\lambda-\alpha_j)(\lambda-\alpha_i)}\, dt\, d\lambda$$

$$+ \lim \frac{1}{2\pi i} \int_{-C_i} \int_a^b \frac{a_i(t) f_i(t) G(x,t,\lambda)}{(\lambda-\alpha_i)^2}\, dt\, d\lambda,$$

for the first of these limits vanishes by Lemma II. Hence the expansion for $f_i(x)$ converges to the value of the final limit above, and this is by Lemma III

$$\tfrac{1}{2}\{f_i(x+0)+f_i(x-0)\}, \text{ when } x \neq a, x \neq b,$$

$$\tfrac{1}{2}\{f_i(a+0)+f_i(b-0)\}, \text{ when } x = a, x = b,$$

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INVARIANT POINTS OF A SURFACE TRANSFORMATION OF GIVEN CLASS*

BY

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1. **Introduction.** The theory of the invariant points of a surface transformation is not only of interest in itself but of some importance in its applications to dynamical and other problems. As such, it has already received a certain amount of attention. Thus, for example, Brouwer has studied the transformations of a sphere and of a projective plane, Kerékjártó the transformations of a sphere and of a plane ring, Nielsen and Brouwer the transformations of the anchor ring and of the non-orientable surface of the same connectivity.† In looking for periodic orbits of dynamical systems, Birkhoff has shown that if a closed orientable surface of genus p be carried into itself by a one-to-one analytic transformation, the difference between the number of directly unstable and other invariant points is a function of the class of the transformation alone and has the value $2p - 2$ when the transformation is of the class of the identity, that is to say, when the transformation can be generated by a continous deformation of the surface.

* Presented to the Society, December 28, 1922.

† L. E. J. Brouwer, *Über eindeutige stetige Transformationen von Flächen in sich*, Mathematische Annalen, vol. 69 (1910), pp. 176–180.

B. von Kerékjártó, *Über die Brouwerschen Fixpunktsätze*, Mathematische Annalen, vol. 80 (1919), pp. 29–32; *Über Transformationen des ebenen Kreisringes*, Mathematische Annalen, vol. 80 (1919), pp. 33–35.

J. Nielsen, *Über die Minimalzahl der Fixpunkte bei den Abbildungstypen der Ringflächen*, Mathematische Annalen, vol. 82 (1921), pp. 83–93.

L. E. J. Brouwer, *Über die Minimalzahl der Fixpunkte bei den Klassen von eindeutigen stetigen Transformationen der Ringflächen*, Mathematische Annalen, vol. 82 (1921), pp. 94–96.

See also the note *Sur quelques applications de l'indice de Kronecker*, by Hadamard appended to Tannery's *Introduction à la Théorie des Fonctions d'une Variable*, and the following:

J. W. Alexander, *On transformations with invariant points*, these Transactions, vol. 23 (1922), pp. 89–95.

G. D. Birkhoff and O. D. Kellogg, *Invariant points in function space*, these Transactions, vol. 23 (1922), pp. 96–115.

G. D. Birkhoff, *Dynamical systems with two degrees of freedom*, these Transactions, vol. 18 (1917), pp. 199–300. In particular, pp. 286 et seq.

I shall consider in this paper a general s-to-1 continuous transformation of a surface into itself and obtain, (§ 6), the explicit relation between the sum of the indices of the invariant points, (§ 2), and the class of the transformation, (§ 3). The generalization from a 1-to-1 to an s-to-1 transformation is only significant when the connectivity of the surface does not exceed 3, (§ 3).

So as not to lengthen the paper unduly, I have assumed with only a few words of explanation some of the simpler topological properties of surfaces. The elements of two-dimensional analysis situs are sufficiently well established by now to permit of their free use without too great insistence on details.

2. **Index of a curve and of an invariant point.** Let Δ be a continuous transformation of the plane or of a portion of the plane such that it carries a finite region R into a finite region $\bar{R}$ and a variable point P of R into a variable point $\bar{P}$ of $\bar{R}$. Then, if P is not an invariant point of the transformation, the angle θ made by the vector $P\bar{P}$ with the positive X-axis is determined in value except for an additive constant of the form $2\tau\pi$ and varies continuously as the point P describes a path in R, provided, of course, that the path avoids the invariant points of Δ, at which θ ceases to have any meaning. Now, if the point P describes a closed path of the above sort in a definite sense, the final value of θ as P returns to its original position will differ from the initial value of θ by an integral multiple τ of 2π, at most. We shall call this number τ the *index* of the closed sensed path with respect to the transformation Δ. This index obviously remains unaltered under any continuous deformation of the path which avoids the invariant points of Δ; consequently, if such a deformation reduces the path to a (non-invariant) point, the index of the path must be zero. We may mention, in passing, that if A be a closed sensed path and $\bar{A}$ its image under a transformation Δ, the index of A with respect to Δ is the same, — not the negative of, — the index of $\bar{A}$ with respect to the inverse transformation Δ^{-1}. Under a change of coördinates in the plane, the index of a sensed curve either remains invariant or changes in sign according as the orientation of the new system is the same as or the reverse of that of the old.

The *index* of an isolated invariant point $F = \bar{F}$ of Δ is defined as the index of the boundary B, positively described, of any simply connected region containing F but no other invariant point of Δ. The index of an invariant point is completely independent of the coördinate system. To be sure, the index of the sensed curve B changes sign when we pass from a right-handed to a left-handed system of coördinates, but the positive direction along B is likewise reversed, so that the two changes exactly compensate one another. From this, it follows that a meaning may be given to the index of an isolated invariant point F of any continuous surface transformation, not necessarily planar, provided the surface undergoing the transformation is free from singularities

in the neighborhood of the fixed point F. For there is no difference, topologically, between the neighborhood of a simple point of a surface and the neighborhood of a point in a plane. It is easy to verify that the index of an invariant point F is the same both with respect to the direct transformation and its inverse Δ^{-1}.

Returning to the plane, let us consider any closed region R' in R bounded by a curve B' and such that there are at most a finite number of invariant points of Δ in R', none of which are on the curve B'. Then the index of the curve B' described in the positive sense evidently measures the sum of the indices of the invariant points within the region R', as may be seen at once by subdividing the region R' into simply connected pieces such that no piece contains more than one invariant point and that no invariant point is on the common boundary of two or more pieces. For, from its definition, the index of B' is equal to the sum of the indices of the boundaries of all the component parts of R' determined by the subdivision.

3. **Surface transformations; class and index of a transformation.** Let S be a closed surface of connectivity $\varkappa$, and let Δ be a continuous transformation which carries a variable point P of S into a variable point $\overline{P}$ of the same surface.* The inverse transformation Δ^{-1} need not be single valued, but we shall assume that, with a finite number of exceptions, its carries a point $\overline{P}$ back into s distinct points of the surface S, which simply means that the transformation Δ is of the so-called s-to-1 type. On the manifold $\overline{S}$ ($\equiv S$) of the transformed points $\overline{P}$, let us form a covering surface S^* of s layers such that the correspondence Δ^* from S to S^* associated with the correspondence Δ from S to $\overline{S}$ is everywhere 1-to-1. Then, by an immediate application of Euler's formula

$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - \varkappa,$$

we may compare the connectivity $\varkappa^*$ of the covering surface S^* with the connectivity $\overline{\varkappa}$ of the surface $\overline{S}$, and so obtain the relation

$$3 - \varkappa^* = s(3 - \overline{\varkappa}) - b, \tag{1}$$

where b is the number of equivalent simple branch points of the covering surface. But the surfaces S^* and $\overline{S}$ are each images of the surface S; therefore, $\varkappa^* = \overline{\varkappa} = \varkappa$, and (1) reduces to

* When the surface is orientable of genus p, we have $p = \frac{1}{2}(\varkappa - 1)$.

$$(s-1)(3-\varkappa)-b = 0. \tag{2}$$

From (2), we conclude that if $\varkappa$ exceeds 3, the transformation must be 1-to-1, and that if $\varkappa$ is equal to 3 the covering surface S^* can have no branch points.

If Δ be an s-to-1 transformation of an oriented surface S into itself, the number $\tau = \pm s$ will be defined as the *index* of the transformation, where the positive or the negative sign is to be chosen before s according as the transformation preserves or reverses orientation. Two transformations of a surface S into itself will be said to belong to the same *class* if and only if they differ from one another by a continuous deformation of the surface into itself. The latter definition applies whether the surface S be orientable or not.

4. **Surfaces of genus** 0. By way of introduction to the general problem, let us first examine an s-to-1 transformation of a closed surface S of genus 0 into itself, although we shall obtain nothing new in this case. We shall assume that the transformation Δ possesses at most a finite number of invariant points. So far as generality is concerned, it is worth bearing in mind that an unrestricted s-to-1 transformation may be regarded as a limiting case of a transformation of type Δ involving a parameter λ.

We represent the surface S by the complex plane and so select the point at infinity that it is neither an invariant point of the transformation Δ nor a point at which the covering surface S^* of § 3 has a branch point. There will then be s distinct finite points $P_1, P_2, \ldots, P_s$ of the plane which are carried to infinity by the transformation Δ. Moreover, if indices be assigned to these points, just as if they were invariant points, it will be found that the index of each is -1 or $+1$, according as the transformation preserves or reverses orientation, because a small positively sensed circle about a point P_i will be carried into a distant curve about infinity encircling the point P_i once in the negative or positive sense according as the transformation preserves orientation or not. In general, there will also be certain fixed points F_j with indices τ_j respectively.

Now, let C be a positively sensed circle with center at the origin and large enough to enclose all the points P_i and F_j. Then, by § 2, the index of C will measure the sum of the indices of all the points P_i and F_j together. Moreover, if the circle C be sufficiently large, its image will be a small curve entirely within C and in the neighborhood of the image of the point at infinity under the transformation Δ. The index of the curve C will, therefore, be unity, and we shall have

$$1 = \sum_j \tau_j \mp s$$

where the sign before s depends upon whether the transformation preserves orientation or not. Thus, if $\tau = \pm s$ be the index of the transformation, (§ 3), we shall have the relation

$$\sum_j \tau_j = 1 + \tau. \tag{3}$$

This is Brouwer's theorem in a generalized form. In the paper referred to in the footnote to § 1, I have given another proof of this theorem, so framed as to generalize to n dimensions and to require no restrictions whatever on the character of the inverse transformation Δ^{-1}.*

The above argument is very closely related to one of the well known proofs of the fundamental theorem of algebra.† In fact, the roots of an equation $f(z) = 0$ are the fixed points of the transformation

$$\bar{z} = z + f(z),$$

and conversely.

5. **Intersection numbers.** To *orient* an orientable surface S, we choose two sensed arcs X and Y crossing one another at a point P of S and arbitrarily agree to say that X makes a positive crossing with Y at P and Y a negative crossing with X. Positive and negative crossings are then determined at all points of S by continuity considerations. Moreover, with any two closed sensed curves A and B of the oriented surface S, there is associated a so-called *intersection number*

$$N(AB) = -N(BA)$$

which can be interpreted to measure the difference between the number of times the curve A makes a positive crossing with the curve B and the number of times it makes a negative crossing with B. This method of interpretation implies, of course, that the curve A crosses the curve B a finite number of times only. However, the number $N(AB)$ may be defined by continuity considerations whether or not this condition is actually realized. The number $N(AB)$ is invariant under continuous deformations of A and B on the surface S.

6. **The fundamental relation.** Let us now consider the case of a 1-to-1 transformation Δ of a closed oriented surface S into itself, where the genus p of S is greater than zero. Through a point C of the surface S, we choose a canonical system of A and B curves

$$(K) \qquad A_1, A_2, \ldots, A_p; B_1, B_2, \ldots, B_p,$$

* See also the very important generalizations to function space by Birkhoff and Kellogg, loc. cit.

† Cf. Osgood, *Lehrbuch der Funktionentheorie*, pp. 219 et seq.

sensed in such a manner that $N(A_i B_i) = -N(B_i A_i) = 1\ (i = 1, 2, \ldots, p)$. Then, under the transformation Δ, the point C is carried into a point $\bar{C}$, and the system (K) into a system

$$(\bar{K}) \qquad \bar{A}_1, \bar{A}_2, \ldots, \bar{A}_p; \bar{B}_1, \bar{B}_2, \ldots, \bar{B}_p.$$

Moreover, $N(\bar{A}_i \bar{B}_i) = -N(\bar{B}_i \bar{A}_i) = 1$, since the transformation preserves orientation. We shall assume that the transformation Δ possesses at most a finite number of invariant points and that the system (K) may be so chosen (i) as to pass through no invariant point and (ii) as to meet the system $(\bar{K})$ in at most a finite number of points at each of which a curve of (K) definitely crosses a curve of $(\bar{K})$. Actually, these conditions are always realized either for Δ itself or for some arbitrarily neighboring transformation Δ' of the same class as Δ. The formula that we are about to prove is the following generalization of (3):

$$(4) \qquad \sum_j \nu_j = 1 + \tau + \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}$$

where the transformation index τ is unity in this particular case, since we are assuming that Δ is 1-to-1 and preserves orientation. If the class of Δ contains the identity,

$$N(B_i \bar{A}_i) = N(\bar{B}_i A_i) = N(B_i A_i) = -1,$$

and (4) reduces to

$$\sum_j \nu_j = 2 - 2p,$$

from which Professor Birkhoff's theorem, mentioned in the introduction, may at once be deduced.

To prove relation (4), we first cut the surface S along the transformed canonical curves $(\bar{K})$ and thereby resolve it into a simple polygonal region of $4p$ sides which we represent by the interior of a circle $\bar{K}$ in the plane. The circumference of the circle $\bar{K}$ now represents the system $(\bar{K})$ itself. It consists of $4p$ arcs following one another in the canonical order

$$\bar{A}_1 \bar{B}_1 \bar{A}_1^{-1} \bar{B}_1^{-1} \cdots \bar{A}_p \bar{B}_p \bar{A}_p^{-1} \bar{B}_p^{-1}$$

as the circle is described in the positive sense. The original canonical system (K) is represented within the circle $\bar{K}$ by a system of arcs terminating either within $\bar{K}$ and at the point C or on the circumference $\bar{K}$ itself. Two or more

of these arcs will always be needed to represent an entire curve of the system (K). The situation is as illustrated by Figure 1 which represents the case of a relatively simple transformation of a surface of genus 2.

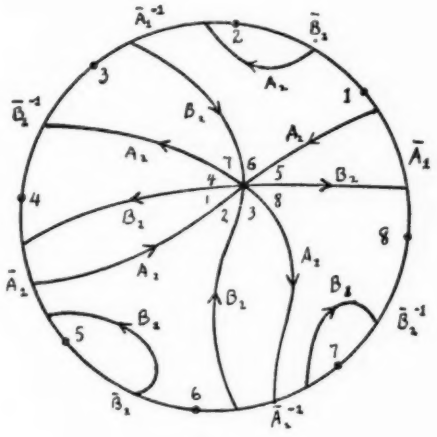

Fig. 1.

Now, it will be observed that the system of arcs (K) subdivides the interior of the circle $\bar{K}$ into a finite number of simply connected regions R_i, and that each of the regions R_i is carried by the transformation Δ into a region $\bar{R}_i$ such that $\bar{R}_i$ is not cut by any of the curves $(\bar{K})$ on S and is therefore represented in the plane by a single connected piece. Hence, in the plane, we may speak about the index of the boundary of each region R_i. Moreover, by § 2, the sum σ_0 of the indices of the boundaries of all the regions R_i, where each boundary is described in the positive sense, is equal to the sum of the indices of the invariant points of Δ:

$$\sigma_0 = \sum_j \tau_j. \tag{5}$$

It is perhaps superfluous to observe that the transformations of the individual regions R_i of the plane do not combine to form one single continuous transformation of the interior of the circle $\bar{K}$ into itself, so that it would be quite meaningless to say that the sum σ_0 was equal to the index of the circle $\bar{K}$. The index of $\bar{K}$ is not defined at all.

We must next identify the sum σ_0 with the right-hand member of (4) in order to establish the validity of that relation. Now, the number $2\pi\sigma_0$ measures

the total variation in the direction θ of a vector $P\bar{P}$ joining a point P within or on the circle $\bar{K}$ to its image $\bar{P}$, as the point P describes the boundaries of all the regions R_i in the positive sense. Moreover, these boundaries consist, in part, of arcs of the system (K) and, in part, of arcs of the circle $\bar{K}$, so that we may write

$$2\pi\sigma_0 = 2\pi\sigma + 2\pi\bar{\sigma},$$

where the two expressions on the right measure the contribution to the total variation of θ coming from the arcs of (K) and of $\bar{K}$ respectively.

To make matters definite, let us first evaluate the contribution to the variation $2\pi\sigma$ from a single curve of (K), such as A_1. In spite of the fact that every arc of A_1 is traversed exactly twice in opposite directions as the boundaries of the regions R_i are described, the total variation of θ over A_1 is not zero, for the image of a point P of A_1 is to be taken on the arc $\bar{A}_1$ of $\bar{K}$ when A_1 is being described in the direct sense, but on the arc $\bar{A}_1^{-1}$ of $\bar{K}$ when A_1 is being described in the reverse sense, corresponding to the fact that the regions on the two sides of an arc of A_1 are mapped upon regions contiguous to $\bar{A}_1$ and $\bar{A}_1^{-1}$ respectively. Let us denote the images of the point P of A_1 on $\bar{A}_1$ and $\bar{A}_1^{-1}$ by $\bar{P}$ and $\bar{P}^{-1}$ respectively. Then, the sum of the variations in direction of $P\bar{P}$ and $P\bar{P}^{-1}$ as P describes A_1 in the direct and reverse senses respectively is equal to the difference of these variations as P is described in the direct sense alone, which difference is merely the variation of the angle $\bar{P}^{-1}P\bar{P}$ denoted by ψ in Figure 2.

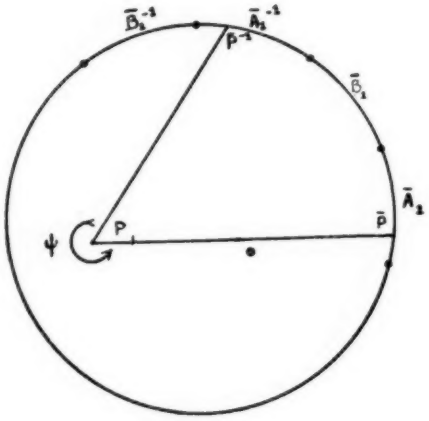

Fig. 2.

Now, as the point P describes the planar representation of the curve A_1, it makes a sudden leap from one point of the circle $\bar{K}$ to another whenever it passes from one arc of A_1 to the next. Moreover, when such a leap occurs, we must prepared for an abrupt variation in the angle ψ, which variation is not to be measured in computing the contribution from A_1 to the variation $2\pi\sigma$. Actually, however, by the elementary properties of angles inscribed in a circle, the angle ψ undergoes no variation at a leap of the point P unless the leap is from one to the other of the two arcs of $\bar{K}$ determined by the points $\bar{P}$ and $\bar{P}^{-1}$, when the angle ψ changes by $\pm\pi$, depending on the direction of the leap. We conclude immediately that ψ decreases by π whenever the point P jumps from the arc $\bar{B}_1$ to the arc $\bar{B}_1^{-1}$ of $\bar{K}$, increases by π whenever P jumps from $\bar{B}_1^{-1}$ to $\bar{B}_1$, but suffers no discontinuity at any other jump.

Let us now denote by α_1 and α_1' the positive angles subtended at the point C by the arcs $\bar{A}_1$ and $\bar{A}_1^{-1}$ of $\bar{K}$ respectively. The total variation of ψ over A_1, including all the jumps, is then $\alpha_1+\alpha_1'$, since the point P starts from and returns to C, whereas the points $\bar{P}$ and $\bar{P}^{-1}$ describe $\bar{A}_1$ and A_1^{-1} respectively in such a way as to augment ψ by α_1 and α_1' respectively. The effective variation of ψ, not counting the jumps, is, therefore,

$$\alpha_1+\alpha_1'-\pi N(A_1\,\bar{B}_1),$$

as may be seen at once by recalling the significance of the intersection number $N(A_1\,B_1)$. In similar fashion, the contribution to the variation $2\pi\sigma$ from the curve A_i is

$$\alpha_i+\alpha_i'-\pi N(A_i\,\bar{B}_i) \tag{6}$$

while that from the curve B_i is

$$\beta_i+\beta_i'+\pi N(B_i\,\bar{A}_i), \tag{7}$$

where the meaning of β_i and β_i' is similar to that of α_1 and α_1'. Thus adding (6) and (7) and summing with respect to i, we finally obtain

$$2\pi\sigma = 2\pi+\pi\sum_i\{N(B_i\,\bar{A}_i)+N(\bar{B}_i\,A_i)\}. \tag{8}$$

In deriving (8), we have, of course, made use of the relations

$$\sum(\alpha_i+\alpha_i'+\beta_i+\beta_i') = 2\pi$$

and

$$-N(A_i\,\bar{B}_i) = N(\bar{B}_i\,A_i).$$

The variation $2\pi\bar{\sigma}$ is computed in similar fashion. Let the system

$$(k) \qquad a_1, a_2, \ldots, a_p; \ b_1, b_2, \ldots, b_p$$

be the image of the system $(\bar{K})$ under the transformation Δ. We then obtain without difficulty the relation

$$(9) \qquad 2\pi\bar{\sigma} = 2\pi + \pi \sum_i \{N(\bar{B}_i a_i) + N(b_i \bar{A}_i)\}.$$

Now,

$$N(\bar{B}_i a_i) = N(B_i \bar{A}_i) \text{ and } N(b_i \bar{A}_i) = (\bar{B}_i A_i).$$

Therefore, from (8) and (9), we obtain

$$(10) \qquad \sigma_0 = \sigma + \bar{\sigma} = 2 + \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

Relation (4) now follows at once from (5) and (10), since $\tau = 1$.

7. Extension to a general transformation. Without going into details, let us merely indicate where the analysis of § 6 varies if it is assumed that the transformation Δ reverses orientation instead of preserving it.

(i) When cuts are made along the system $(\bar{K})$ and the resulting polygon is mapped in the plane in such a way that a positive crossing on the surface S is represented by a positive crossing in the plane, we find that, in this case, the arcs of the circle $\bar{K}$ follow one another in the cyclic order

$$\bar{B}_p \bar{A}_p \bar{B}_p^{-1} \bar{A}_p^{-1} \cdots \bar{B}_1 \bar{A}_1 \bar{B}_1^{-1} \bar{A}_1^{-1}$$

as $\bar{K}$ is described in the positive sense.

(ii) As a result of the above change, the contributions to $2\pi\sigma$ from the curves A_i and B_i become

$$-\alpha_i - \alpha_i' - \pi N(A_i \bar{B}_i)$$

and

$$-\beta_i - \beta_i' + \pi N(B_i \bar{A}_i)$$

respectively, giving

$$2\pi\sigma = -2\pi + \pi \sum_i \{N(B_i \bar{A}_i) + N(\bar{B}_i A_i)\}.$$

(iii) The expression for $2\pi\bar{\sigma}$ is now

$$(9') \qquad 2\pi\bar{\sigma} = 2\pi - \pi \sum \{N(\bar{B}_i a_i) + N(b_i \bar{A}_i)\}.$$

However, since the transformation reverses orientation,

$$N(\bar{B}_i a_i) = -N(B_i \bar{A}_i), \quad N(b_i \bar{A}_i) = -N(\bar{B}_i A_i),$$

so that

$$2\pi\bar{\sigma} = 2\pi + \pi\sum\{N(B_i\bar{A}_i) + N(\bar{B}_i A_i)\}.$$

Thus, in place of (10), we have

$$\sigma_0 = \sum_i\{N(B_i\bar{A}_i) + N(\bar{B}_i A_i)\}. \tag{10'}$$

Thus, by (5) and (10'), relation (4) is again verified, since τ is now equal to -1.

To complete the discussion for an orientable surface S, there remains to be considered the case of an s-to-1 transformation, where s is greater than unity. However, by § 3, a transformation of this sort can only exist if the genus p of the surface is unity,

$$p = \tfrac{1}{2}(\varkappa - 1) = 1,$$

so that we are back to the case already discussed by Nielsen and Brouwer.* It will, therefore, be sufficient to verify relation (4) in terms of the results obtained by these mathematicians. We merely remark, in passing, that the method of § 6 may readily be extended to cover this case also. We take as the system $(\bar{K})$, along which the cuts are made, a pair of canonical curves $\bar{A}$ and $\bar{B}$, each of which is the image of s distinct curves $A^{(1)}, \ldots, A^{(s)}$ and $B^{(1)}, \ldots, B^{(s)}$. These $2s$ curves form the system (K). The system (k), however, consists of two curves a and b only. Proceeding as before, we find

$$2\pi\sigma = 2\pi\tau + \pi\{N(B\bar{A}) + N(\bar{B}A)\},$$

$$2\pi\bar{\sigma} = 2\pi + \pi\{N(B\bar{A}) + N(\bar{B}A)\},$$

therefore,

$$\sum_j \tau_j = 1 + \tau + N(B\bar{A}) + N(\bar{B}A). \tag{4'}$$

This is nothing but relation (4) for the special case under consideration.

The results of Nielsen and Brouwer may be put in the following form. If the curves A and B transform into

$$\bar{A} = A^m B^n \text{ and } \bar{B} = A^p B^q$$

respectively, then

$$\sum\tau_q = mq - np - m - q + 1. \tag{4''}$$

Relations (4') and (4'') are equivalent, however, for

* Loc. cit., Mathematische Annalen, vol. 82.

$$N(B\bar{A}) = -m, \qquad N(\bar{B}A) = -q,$$

$$r = mq - np.$$

8. **Non-orientable surfaces.** The case of a non-orientable or so-called "one-sided" surface is much less important in the applications and may be dealt with in summary fashion as follows.

With every non-orientable surface T, there is associated a unique orientable 2-sheeted covering surface S such that each point P_s of S corresponds to the point P_t of T which it covers associated with one of the two possible choices of an indicatrix at P_t. Moreover, every transformation Δ of the surface T into itself determines two transformations $\Delta^{(1)}$ and $\Delta^{(2)}$ of the covering surface S such that the two points $P_s^{(1)}$ and $P_s^{(2)}$ of S covering a point P_t of T are transformed respectively into the two points covering the image of P_t under Δ. The two transformations $\Delta^{(1)}$ and $\Delta^{(2)}$ differ merely by a permutation of the sheets of S; one of them preserves, the other reverses orientation. Finally, every invariant point of Δ corresponds to an invariant point of one but not both of the transformations $\Delta^{(1)}$ and $\Delta^{(2)}$. Thus, the sum of the indices of the fixed points of Δ is expressed as the sum of a similar pair of sums for $\Delta^{(1)}$ and $\Delta^{(2)}$, and we are back to the orientable case already considered.

9. **Boundaries.** In the applications to dynamics,* it is useful to consider the transformations Δ of an orientable surface S of given genus but bounded by d simple closed curves, each of which is transformed into itself in such a way as to leave no point invariant. Obviously, each of these curves plays the rôle of an invariant point of index unity. For if the surface S be closed by the addition of d simply connected pieces bounded by the d curves respectively, it will be possible to extend the definition of the transformation Δ to the added pieces in such a way that within each piece there will be a single invariant point of index unity. It would also be possible to consider the case where two or more boundaries were interchanged. The presence of such boundaries would not affect the fundamental formula at all. To sum up, if the surface S contains boundaries of which d are transformed into themselves without invariant point,

$$\sum_j r_j + d = 1 + r + \sum_{i=1}^{r} \{N(B_i\bar{A}_i) + N(\bar{B}_i A_i)\}.$$

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* Cf. Birkhoff, loc. cit., pp. 290, et seq.

APPLICATIONS OF ANALYSIS TO THE ARITHMETIC OF HIGHER FORMS*

BY

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1. The classical applications of elliptic functions to the theory of numbers are confined to a limited range of problems concerning enumerative functions in a narrow strip of additive arithmetic. The historically first instance is typical of all. In it there is determined more readily by means of elliptic functions than by any other method the *number* of representations of an integer as a *sum* of four *squares*. Class number formulas have essentially the same characteristics: the range is everywhere bounded by enumerative theorems on representations in homogeneous forms of the second degree.

2. As a preliminary extension it was shown† that the range can be broadened to include paraphrases involving L-functions of any parity. In this extension the associated forms are still of the second degree. The simplest cases of the paraphrases thus obtained are enumerative theorems of the kind just described. For if $f(x)$ is of parity $p(1\,|\,0)$ we can take as a very special case $f(x) = 1$ for all values of x.

3. The wider range in § 2 can again be extended indefinitely in a new direction. It will be shown that we can always find general relations (generalized paraphrases) concerning L-functions of any parity and of any order in which the arguments of the L-functions are linear functions of the indeterminates which represent an arbitrary constant integer in forms of degree n_1 and order n_1' with at least one of the coefficients in the forms an arbitrary constant integer, where $n_1 > n$, $n_1' > n'$, and n, n' are any preassigned integers > 0. We pass therefore from the preliminary paraphrases of § 2 in which the associated forms are of the second degree to like paraphrases in which the forms are of any degree.

4. The extended paraphrases of § 3 can be yet further generalized. We shall show that it is sufficient in forming the arguments of the L-functions to include only those values of the indeterminates which are such that arbitrary constant even positive powers of them lie within any preassigned limits.

* Presented to the Society, San Francisco Section, October 21, 1922.

† These Transactions, vol. 22 (1921), p. 1. We presuppose §§ 1, 2, 5, 6, 8, 9 of that paper.

5. It will be shown that the extension in § 4 includes as a special case that in § 3, which in turn includes the extension in § 2, and this again includes the type of result outlined in § 1. Although we have referred above only to elliptic functions it will be evident (cf. § 6 of the former paper) that a precisely similar analysis applies to relations between the multiple theta functions and that the presence of periodic quotients in such relations is not a prerequisite for the applications of the method.

To preserve the continuity of the discussion we omit all illustrations, a detailed set of which will be given in another paper.

6. Let $\lambda, \mu, \cdots, \nu$ denote arbitrary odd constant integers > 0, and $a, b, \cdots, c$ arbitrary constant integers $\neq 0$. Let

$$\varphi_{\alpha p}(w) = 1, \quad (\alpha, p, w) \equiv (\lambda, a, x), \quad (\mu, b, y), \cdots, (\nu, c, z)$$

if w is of the form $p\,t^{\alpha}$ where t is an integer $\gtreqless 0$, and in all other cases let $\varphi_{\alpha p}(w) = 0$. Since α is odd it is evident that $\varphi_{\alpha p}(w)$ is an even function of w. It is also clear that the similarly defined function $\varphi_{\beta p}(w)$ in which β is not odd is neither an even nor an odd function of w. Let $f \equiv f(x, y, \cdots, z)$ have in addition to the property

(*A*) f is single valued whenever its arguments $x, y, \cdots, z$ are simultaneously integers $\gtreqless 0$;

one (and necessarily only one) of the following properties:

(*B*) f is an even function of each of its arguments;

(*C*) f is an odd function of each of its arguments;

(*B'*) f is an even function of all of its arguments,

$$f(x, y, \cdots, z) = f(-x, -y, \cdots, -z);$$

(*C'*) f is an odd function of all of its arguments,

$$f(x, y, \cdots, z) = -f(-x, -y, \cdots, -z).$$

Beyond the specific pair of these properties which f possesses it is wholly arbitrary. The last is essential in what follows.

If now f has the property A and any one A' of B, C, B', C', so also has

$$\varphi_{\lambda a}(x)\,\varphi_{\mu b}(y)\cdots\varphi_{\nu c}(z)\,f\left(\sqrt[\lambda]{\frac{x}{a}},\ \sqrt[\mu]{\frac{y}{b}}, \cdots, \sqrt[\nu]{\frac{z}{c}}\right),$$

considered as a function of the arguments $x, y, \cdots, z$, the same pair A, A' of properties.

7. Consider all the sets $(x, y, \cdots, z)$ $(i = 1, 2, \cdots)$ of integers satisfying a given set S of conditions. Among these some (or none) will also satisfy the conditions

$$x_i = a\xi_i^{\lambda}, \qquad y_i = b\eta_i^{\mu}, \qquad \cdots \qquad z_i = c\zeta_i^{\nu}, \tag{1}$$

where the only restriction upon $\xi_i, \eta_i, \cdots, \zeta_i$ is that they shall be integers $\geqq 0$. Let $(\xi_i, \eta_i, \cdots, \zeta_i)$ $(i = 1, 2, \cdots)$ be all the sets of values of the $\xi_i, \eta_i, \cdots, \zeta_i$ obtained in this way.

Suppose we have found the relation

$$\sum_i k_i f(x_i, y_i, \cdots, z_i) = 0, \tag{2}$$

in which the k_i are numerical constants, the sum extends to all sets $(x_i, y_i, \cdots, z_i)$, and f has the properties A, A'. Since f is unrestricted beyond A, A', (2) implies

$$\sum_i k_i \varphi_{\lambda a}(x_i)\ \varphi_{\mu b}(y_i) \cdots \varphi_{\nu c}(z_i) f\left(\sqrt[\lambda]{\frac{x_i}{a}}, \sqrt[\mu]{\frac{y_i}{b}}, \cdots, \sqrt[\nu]{\frac{z_i}{c}}\right) = 0,$$

which by (1) and the definition of φ is equivalent to

$$\sum_i k_i f(\xi_i, \eta_i, \cdots, \zeta_i) = 0, \tag{3}$$

the sum extending to all sets $(\xi_i, \eta_i, \cdots, \zeta_i)$. Only these sets appearing in (3), we may use (1) to replace the set S of conditions upon the $(x_i, y_i, \cdots, z_i)$ by a set S' upon the $(\xi_i, \eta_i, \cdots, \zeta_i)$. We may then consider (3) directly in relation to S' without the intermediary S. If (2) is a paraphrase of the kind cited in § 2 for L-functions of one or other of the parities $p(1, 1, \cdots, 1|0)$, $p(0|1, 1, \cdots, 1)$, $p(n|0)$, $p(0|n)$, where n is the number of independent variables in the set $(x, y, \cdots, z)$ and the unit in the first two symbols is repeated n times, then (3) is a result of the sort described in § 3 for the same L-functions.

8. To reach the further extension summarized in § 4 we now proceed as follows. We are to select only the $(\xi_i, \eta_i, \cdots, \zeta_i)$ in the limited region

$$r \leqq \xi_i^g \leqq \varrho, \qquad s \leqq \eta_i^h \leqq \sigma, \qquad \cdots, \qquad t \leqq \zeta_i^k \leqq \tau, \tag{4}$$

where $r, \varrho, s, \sigma, \cdots, t, \tau$ are arbitrary constants (but consistent with the respective inequalities) and $g, h, \cdots, k$ are arbitrary constant even integers > 0.

Let the function $\theta_{pj\pi}(x) = 1$ if $p \leqq x^j \leqq \pi$, where p, π are numerical constants, and in all other cases let this function vanish. Then if j is an even integer > 0, $\theta_{pj\pi}(x)$ is an even function of x, while if j is not an even integer, the function is neither even nor odd.

Hence, by an argument precisely similar to that in § 7, if $(\xi_i', \eta_i', \cdots, \zeta_i')$ $(i = 1, 2, \cdots)$ are all the sets $(\xi_i, \eta_i, \cdots, \zeta_i)$ which satisfy in addition to S' the conditions (4), we may replace (3) by

$$\sum_i k_i f(\xi_i', \eta_i', \cdots, \zeta_i') = 0. \tag{5}$$

Finally then we combine (4) and S' into one set S'' of conditions, and as in § 7 consider (5) directly in relation to S''. The result is as described in § 4 for L-functions of the parities indicated in § 7.

9. For the special values $(r, g, \varrho), (s, h, \sigma), \cdots, (t, k, \tau) = (0, 2, \infty)$ (5) becomes (3), and when further in (1) we put

$$(a, \lambda) = (b, \mu), \cdots, (c, \nu) = (1, 1),$$

(3) degenerates to (2). Hence (5) includes both (3), (2).

10. We say that S'' has been obtained from S by extension. In extending a given set of conditions upon a particular system of indeterminates we therefore subject the system to the further conditions (1), (4). Now a paraphrase of the preliminary type (former paper, p. 5) refers to a separation in which no indeterminate occurs to a power higher than the second. By extension as above we replace such a separation by an extended set of conditions in which:

(i) Some (or all) of the indeterminates occur to arbitrary odd constant powers which may have the same or different exponents;

(ii) The coefficients of the terms involving these powers are arbitrary constant integers;

(iii) The ranges of variation of the indeterminates in (i) are so restricted that an arbitrary constant even power of each indeterminate lies within any preassigned limits.

The result of this extension is called an *extended separation*. As shown above this extension includes the kind of the former paper as one of its simplest cases.

11. If n is as in § 7 the number of independent variables in the set $(x, y, \cdots, z)$ the function f in (2), (3), (5) is, by B, C, B', C' of § 6, of one or other of the parities indicated in § 7. The generalization of all results thus far obtained follows immediately upon repeated application of the

argument in §§ 7, 8, 9, and we can pass at once from a paraphrase of the form (2) in which f is now an L-function of parity

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s)$$

where the a's and b's are any integers > 0, to equations (3) and (5) in which now f is of the same parity and the summation in (5) is performed with respect to an extended separation. In this, f retains its complete generality beyond the specific equations of parity which it satisfies.

In the special case previously discussed we considered L-functions subject to restrictions beyond their parity conditions. The restrictions were chiefly of the type that the function changes sign or remains invariant under permutation of some or all of its variables. Clearly all such results can be carried over to the extended case.

12. The extended principle of paraphrase can now be stated.

Any identity in elliptic, abelian or theta functions, provided only that it contains the arguments of the functions and is not merely an identity between constants, implies and is implied by an identity between general L-functions integrated over an extended separation.

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ON THE SECOND DERIVATIVES OF AN EXTREMAL-INTEGRAL WITH AN APPLICATION TO A PROBLEM WITH VARIABLE END POINTS

(SUPPLEMENTARY PAPER)

BY

ARNOLD DRESDEN

In a paper with the above title published in these Transactions* there appeared a statement that "$B(x)$ becomes infinite as x approaches x_1'". In a letter, dated October 18, 1921, Professor Hans Hahn called attention to the fact that the proof of this statement, as implied by the context, although not explicitly mentioned, was insufficient, because it depended upon the non-vanishing of the quadratic form $\sum_{ij} \beta_{ij}\, c_i\, c_j$, which had been proved to be definite by the aid of the fact that $Z(x) \neq 0$, and this condition does not hold at x_1'. It is the purpose of this note to supply a proof of the statement quoted above, upon which the validity of the concluding theorem of the earlier paper depends. I acknowledge my indebtedness to Professor Hahn for having pointed out the need for this supplementary proof.

Starting with the definition of $B(x)$ as given on page 435 and putting $\tau_{ij}(x) = \sum_k R_{ik}(x_1)\, Z^{(kj)}(x)$, we have $B(x) = \sum_{ij} \tau_{ij}\, c_i\, c_j / Z$. We shall show first that not every τ_{ij} can vanish at x_1' of the same order as $Z(x)$. For, suppose that $Z(x) = (x - x_1')^s\, Z_1(x)$, $Z_1(x_1') \neq 0$. If now *every* $\tau_{ij}(x)$ had the factor $(x - x_1')^s$, then, since $|R_{ik}(x_1)| \neq 0$† it would follow that *every* $Z^{(jk)}(x)$ had it also. From the definition of the functions $Z^{(jk)}(x)$, we derive the following system of equations:

$$
\begin{aligned}
Z^{(kj)}(x) &= \sum_l \zeta_{kl}(x)\, Z_{jl}(x), \\
\delta_{kj}\, Z(x) &= \sum_l \zeta_{n+k,l}(x)\, Z_{jl}(x),
\end{aligned}
\tag{1}
$$

in which Z_{jl} designates the cofactor of $\zeta_{n+j,l}$ in the determinant Z, and δ_{kj} is the familiar Kronecker symbol. For every fixed j, we have here a system

* Vol. 17 (1916), p. 436.

† Loc. cit., p. 434, equation (22); also Bolza, *Vorlesungen über Variationsrechnung*, p. 609.

of $2n$ equations. If now every $Z^{(kj)}(x_1') = 0$, we can form from (1) a set of $\binom{2n}{n}$ systems of n linear homogeneous equations in $Z_{jl}(x_1')$, $l = 1, \ldots, n$, for every value of j from 1 to n. The determinants of these systems would include every nth order determinant formed from the matrix

$$\left\| \begin{matrix} \zeta_{1,1}(x), \ldots, \zeta_{2n,1}(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \zeta_{1,3n}(x), \ldots, \zeta_{2n,n}(x) \end{matrix} \right\| \tag{2}$$

for the value x_1'. It would therefore follow that either *every* $Z_{jl}(x_1') = 0$ or the matrix (2) has rank less than n. In the former case, equations (1) could be divided through by $x - x_1'$ and the argument repeated. This may be continued until we reach the conclusion that either every $Z_{jl}(x)$ has the factor $(x - x_1')^s$ or else the rank of the matrix is less than n. The former of these alternatives contradicts the hypothesis that $Z(x)$ has the factor $(x - x_1')^s$ but not the factor $(x - x_1')^{s+l}$. The latter alternative would carry with it the consequence that the $2n$-rowed determinant obtained by adjoining to the matrix (2) n rows consisting of the derivatives of the functions appearing in (2) would vanish at x_1'. But since this determinant is the determinant of a fundamental system of solutions of Jacobi's equations,* it does not vanish. We conclude that not every $\tau_{ij}(x)$ has the factor $(x - x_1')^s$.

Next we observe that the matrix T of the quadratic form $\sum_{ij} \tau_{ij} c_i c_j$ is equal to $|R_{ik}(x_1)| Z_0(x) Y(x)$, where $Z_0(x)$ is the determinant formed from the first n columns of (2) and $Y(x)$ is the adjoint of $Z(x)$. Moreover, a k-rowed minor of T is equal to a sum of k-rowed minors of Y, each multiplied by polynomials in $R_{ik}(x_1)$ and the elements of $Z_0(x)$. If now $Z(x_1')$ is of rank $n-1$, then $Y(x_1')$ is of rank 0 or 1 and consequently $T(x_1')$ is of rank 0 or 1.† If $T(x_1')$ were of rank 0, every $\tau_{ij}(x_1')$ would vanish, which would contradict the result obtained above. Hence $T(x_1')$ is of rank 1 and hence $\sum_{ij} \tau_{ij}(x_1') c_i c_j$ is different from 0, since moreover $\sum_i c_i^2 \neq 0$.‡ For the case that the rank of $Z(x_1')$ is $n - 1$ our statement has therefore been proved.

Suppose now that the rank of $Z(x_1')$ is $n - s\,(s > 1)$. The rank of every $Z_{jl}(x_1')$ is then at most $n - s$. In this case $Z(x) = (x - x_1')^s Z_1(x)$, $Z_{lj}(x) = (x - x_1')^{s-1} Z_{lj_1}(x)$, where not *every* $Z_{lj_1}(x_1')$ is equal to 0. We have then $B(x) = \sum_{ij} \tau_{ij_1}(x) c_i c_j / Z_1(x)(x - x_1')$, where $\tau_{ij_1}(x) = \sum_{kl} R_{ik}(x_1) \zeta_{kl}(x) Z_{jl_1}(x)$ and $\tau_{ij}(x) = (x - x_1')^{s-1} \tau_{ij_1}(x)$ and there-

* Loc. cit., p. 435.

† See Kowalewski, *Determinantentheorie*, p. 81.

‡ Loc. cit., p. 434.

fore not every $\tau_{ij_1}(x_1') = 0$. Now we can set up a determinant of which $Z_{jl_1}(x)$ are the cofactors. The elements of this determinant are given by the formula $z_{jl}(x) = \xi_{jl}(x)/|Z_{jl_1}(x)|^{\frac{n-2}{n-1}}$, where ξ_{jl} is the cofactor of Z_{jl_1} in $|Z_{jl_1}|$. We find now that $|z_{jl}(x)| = Z_1(x)(x - x_1')^{\frac{n-s}{n-1}}$, so that $|z_{jl}(x_1')| = 0$. Consequently the adjoint of this determinant, viz. $|Z_{jl_1}(x_1')|$, is of rank 0 or 1. Furthermore the matrix T_1 of the quadratic form $\sum_{ij} \tau_{ij_1} c_i c_j$ is equal to $|R_{ik}(x_1)| \cdot Z_0(x) \cdot |Z_{jl_1}(x)|$ and is therefore of rank 0 or 1 at x_1'. We proceed now exactly as in the case $s = 1$, treated above, and conclude that $\sum_{ij} \tau_{ij_1}(x_1') c_i c_j$ is different from 0. Hence we have also in this case the result that $\lim_{x \to x_1} B(x) = \infty$.

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ABSTRACT GROUP DEFINITIONS AND APPLICATIONS*

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INTRODUCTION

One of the fundamental problems of group theory is the formulation of abstract definitions for a given group. These definitions to be ideal must exclude redundant conditions for determining the given group, and they should be simple and tend to indicate some of the properties of the group. Abstract definitions for the substitution groups of degree eight and lower degrees have been worked out, and many groups of low order have been abstractly defined. Kronecker† gave the first satisfactory proof that an abelian group is the direct product of its independent cyclic subgroups, and hence its definition readily follows. Accordingly, the problem of abstract definition has to do chiefly with non-abelian groups. A few systems of non-abelian groups composed of a finite number of groups, as well as a few systems composed of an infinite number of groups, have been abstractly defined.

The notion of the cyclic group and its mode of generation have long been known through its association with periodic or cyclic events. The symbolic definition of groups other than cyclic groups was made by Cauchy‡, in 1845, but whether he thought of them as definitions of abstract groups is questionable, as he was then working with substitution groups. In his work of 1846, however, he seemed to have grasped the concept of abstract groups§. In 1849, Bravais‖ stated the conditions defining the groups of movements of the regular polyhedrons. Cayley¶, in 1854, studied the properties of groups all of whose operators satisfy the condition $\theta^n = 1$, in connection with the roots of unity. He gave abstract definitions for the groups of orders 4 and 6, and for a group of each of the orders 18 and 27. The group of order 27 is defined with redundant conditions, and is one of the groups of the system $s_1^3 = s_2^3 = (s_1\, s_2)^3$

* Presented to the Society, December 29, 1922.

† Berliner Monatsberichte, 1870, p. 881.

‡ *Collected Works*, ser. I, vol. IX, pp. 327, 364.

§ *Collected Works*, ser. I, vol. X; Miller, Bibliotheca Mathematica, 1910, p. 10.

‖ Journal de Mathématiques, ser. 1, vol. 1 (1849), pp. 141—180.

¶ Philosophical Magazine, vol. 7 (1854), pp. 40—47; *Collected Papers*, vol. II, p. 123.

$= (s_1 s_2^2)^\alpha = 1$ discussed in this paper. In 1859* he gave a complete list with definitions of the groups of order 8, and discussed the definition

$$\alpha^m = \beta^n = 1, \qquad \alpha\beta = \beta\alpha^s,$$

in which he says the condition $s^n \equiv 1$, mod m, is implied, and from which he shows that several distinct groups of the same order may be determined, depending on the number of solutions of the modular equation, and the order of the group so determined is mn. He also stated that $\beta^2 = \gamma^2 = (\beta\gamma)^5 = 1$ defines a group of order 10, and that such a group exists for any even number $2p$, where β and γ are both of order 2. Hence Cayley explicitly defines the dihedral group. Hamilton†, in 1856, stated that two non-commutative operators satisfying

$$s_1^2 = s_2^3 = (s_1 s_2)^k = 1 \qquad (k = 3, 4, 5)$$

define the groups of movements of the regular polyhedrons.

In this paper groups are considered which are generated by two operators t_1 and t_2 whose product is of order two or three. While this imposes a condition on the operators, still it is not sufficient, except in a few cases, to determine a group of finite order. Miller‡ has shown that two operators of any given orders may have their product of any desired order and that the group generated is completely defined only when two of the operators are of order two, or when one operator is of order two, the second operator is of order three, and their product is of order 3, 4 or 5. Otherwise generators may be found such that the group generated may be any one of some infinite system of groups. Setting

$$t_1^\alpha = t_2^\beta = (t_1 t_2)^2 = 1, \quad \text{or } t_1^\alpha = t_2^\beta = (t_1 t_2)^3 = 1,$$

some general properties are found, and upon giving special values to α and β the remaining conditions that are necessary and sufficient to define a group of finite order are determined. The investigation is made by means of the orders of products of powers of t_1 and t_2. In general the orders of certain products will be arbitrary, and these arbitrary orders may be considered as parameters and the order of the group may then be considered as a function of these parameters. The dihedral and dicyclic groups afford well known examples of this idea.

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* Philosophical Magazine, vol. 18 (1859), pp. 34—37, *Collected Papers*, vol. IV, pp. 88—91.

† Proceedings of the Royal Irish Academy, vol. 6 (1853—57) and Philosophical Magazine, vol. 12 (1856), p. 446.

‡ American Journal of Mathematics, vol. 22 (1900), pp. 185—190.

The theory following has to do with groups whose order is the *maximum*, that is, the generators must not only *satisfy* but also *fulfil* the conditions imposed by the definitions. When $t_1^\alpha = 1$ is written, it is to be understood, unless otherwise explicitly stated, that t_1 is of order α and not of lower order.

I. Preliminary lemmas

Frequent use will be made of the following well known facts: (1) If $(t_1\, t_2)^n = 1$, then $(t_2\, t_1)^n = 1$; (2) If $(t_1\, t_2\, t_3 \cdots t_n)^n = 1$, then $(t_2\, t_3 \cdots t_n\, t_1)^n = \cdots = (t_n\, t_1\, t_2\, t_3 \cdots)^n = 1$.

LEMMA I. *If* $t_1^\alpha = t_2^\beta = (t_1\, t_2)^2 = 1$, *then*

(1) $(t_1\, t_2^2)^\alpha = 1$, $\quad (t_1^2\, t_2)^\beta = 1$;

(2) $(t_1^\delta\, t_2)^\Phi = (t_1^{\delta-2}\, t_2^{-1})^\Phi = 1$, *where* Φ *is the order of* $t_1^\delta\, t_2$;

(3) $(t_1^\delta\, t_2)^\varepsilon = (t_1\, t_2^{-1})^\varepsilon = (t_1^{-1}\, t_2^{-3})^\varepsilon = 1$, *where* ε *is the order of* $t_1^\delta\, t_2$.

Proof:

(1) $t_1\, t_2^2 = t_1\, t_2 \cdot t_2 = t_2^{-1}\, t_1^{-1}\, t_2$, which is of order α;

$t_1^2\, t_2 = t_1 \cdot t_1\, t_2 = t_1\, t_2^{-1}\, t_1^{-1}$, which is of order β.

(2) $(t_1\, t_2)^{-1}\, (t_1^\delta\, t_2)\, t_1\, t_2 = t_2^{-1}\, t_1^{\delta-2}$, which is therefore of the same order as $t_1^\delta\, t_2$.

(3) By (2), $t_1^3\, t_2$ and $t_2^{-1}\, t_1$ are of the same order and the transform of $t_2^{-1}\, t_1$ by t_2 is $t_2^{-3}\, t_1^{-1}$.

LEMMA II. *If* $t_1^\alpha = t_2^\beta = (t_1\, t_2)^2 = (t_1^2\, t_2^2)^\delta = 1$, *then* $t_1\, t_2$ *transforms* $t_1^2\, t_2^2$ *into its inverse and* $t_1\, t_2$ *and* $t_1^2\, t_2^2$ *generate a dihedral group of order* $2\,\delta$.

Proof: $(t_1\, t_2)^{-1}\, t_1^2\, t_2^2\, t_1\, t_2 = t_2^{-1}\, t_1\, t_2\, t_1^{-1} = t_2^{-2}\, t_1^{-2}$, and the product $t_1\, t_2 \cdot t_1^2\, t_2^2$ is of order 2, and hence $t_1\, t_2$ and $t_1^2\, t_2^2$ define a dihedral group of order $2\,\delta$.*

LEMMA III. *If* $t_1^\alpha = t_2^\beta = (t_1\, t_2)^2 = 1$, *then* $t_1^2\, t_2^2$, $t_2^2\, t_1^2$, $t_1\, t_2^2\, t_1$, *and* $t_2\, t_1^2\, t_2$ *are commutators, and* $t_1\, t_2^2\, t_1$ *and* $t_2\, t_1^2\, t_2$ *are inverses.*

Proof:

$$t_1^2\, t_2^2 = t_1\, t_1\, t_2\, t_2 = t_1\, t_2^{-1}\, t_1^{-1}\, t_2; \qquad t_2^2\, t_1^2 = t_2\, t_1^{-1}\, t_2^{-1}\, t_1;$$

$$t_1\, t_2^2\, t_1 = t_1\, t_2\, t_2\, t_1 = t_1\, t_2\, t_1^{-1}\, t_2^{-1}; \qquad t_2\, t_1^2\, t_2 = t_2\, t_1\, t_2^{-1}\, t_1^{-1};$$

$$t_1\, t_2^2\, t_1 \cdot t_2\, t_1^2\, t_2 = t_1\, t_2\, t_1\, t_2 = 1.$$

* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 143.

LEMMA IV. *If* $(t_1 t_2)^3 = 1$, *then* $t_2 t_1^2$ *and* $t_1 t_2^2$ *have the same order.*
Proof:

$$t_1 t_2 t_1 = t_2^{-1} t_1^{-1} t_2^{-1} \text{ by hypothesis;}$$

$$t_1 t_2 t_1 = t_1 \cdot t_2 t_1^2 \cdot t_1^{-1}, \quad \text{and} \quad t_2^{-1} t_1^{-1} t_2^{-1} = t_2^{-1} \cdot t_1^{-1} t_2^{-2} \cdot t_2.$$

II. THE INFINITE SYSTEM OF NON-ABELIAN GROUPS DEFINED BY $t_1^4 = t_2^4 = (t_1 t_2)^2 = 1$ AND ONE ADDITIONAL CONDITION

Consider first the conditions

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = 1, \qquad t_1 t_2 \neq t_2 t_1.$$

The orders of the products $t_1 t_2^2$ and $t_1^2 t_2$ are known by Lemma 1, so that we have the following table of orders of products:

1	t_2	t_2^2	t_2^3
t_1	2	4	α
t_1^2	4	β	4
t_1^3	α	4	2

which implies that the order of $t_1 t_2^3$ or $t_1^2 t_2^2$ is arbitrary. It will now be shown that α and β are not independent. The operators $t_1 t_2^3$ and $t_1^3 t_2$ are commutative and their product is $t_1^2 t_2^2$, that is, these two operators whose orders equal α have a product of order β. Then

$$(t_1^2 t_2^2)^\beta = (t_1 t_2^3 \cdot t_1^3 t_2)^\beta = (t_1 t_2^3)^\beta \cdot (t_1^3 t_2)^\beta = 1,$$

and hence the order of $t_1^2 t_2^2$ is a divisor of α. Suppose now that β is less than α. Then

$$(t_1 t_2^3)^\beta = (t_1^3 t_2)^{-\beta}.$$

Also

$$(t_1 t_2^3)^{-1} (t_1^3 t_2) = t_2 t_1^3 t_1^3 t_2 = t_2 t_1^2 t_2,$$

and this is of order β, since

$$t_2^{-1} (t_2^2 t_1^2) t_2 = t_2 t_1^2 t_2.$$

Hence

$$[(t_1 t_2^3)^{-1} \cdot t_1^3 t_2]^\beta = (t_1 t_2^3)^{-\beta} \cdot (t_1^3 t_2)^\beta = 1,$$

since $t_1 t_2^3$ and $t_1^3 t_2$ are commutative. Therefore

$$(t_1 t_2^3)^\beta = (t_1^3 t_2)^\beta.$$

But $(t_1 t_2^3)^\beta = (t_1^3 t_2)^{-\beta}$, whence $(t_1^3 t_2)^{-\beta} = (t_1^3 t_2)^\beta$ and $(t_1^3 t_2)^{2\beta} = 1$, and therefore $\alpha = 2\beta$. Hence, if β is less than α, the only value it may have is $\alpha/2$.

The operators $t_1 t_2^3$ and $t_1^3 t_2$, being commutative, generate an abelian group of order α^2 at most. Call this abelian group H. Then H is invariant under both t_1 and t_2 since

$$t_1^{-1} t_1 t_2^3 t_1 = t_2^3 t_1, \qquad t_1^{-1} t_1^3 t_2 t_1 = t_1^{-3} t_2^3 = t_1 t_2^3,$$

$$t_2^{-1} t_1 t_2^3 t_2 = t_2^3 t_1, \qquad t_2^{-1} t_1^3 t_2 t_2 = t_1 t_2^3.$$

If H contains either t_1 or t_2 it contains both, and then t_1 and t_2 are commutative, contrary to hypothesis. Hence t_1 and H generate a group of order $4\alpha^2$ at most.* Call this group G.

It has already been shown that β is a divisor of α and that the only value other than α that it can have is $\alpha/2$. In this latter case the order of H is not greater than $\alpha^2/2$, and t_1 and H generate a group whose order is not greater than $2\alpha^2$. It is evident that t_1 and H generate the same group as t_1 and t_2, since t_1 and t_2 transform each of the generators of H into the inverse of the other generator. Hence the order of G is not greater than $4\alpha^2$ or $2\alpha^2$, according as $\beta = \alpha$ or $\beta = \alpha/2$. The preceding results may be summarized in the following two abstract definitions:

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1,$$

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\beta = 1.$$

The first definition defines a group of order not greater than $4\alpha^2$, and the second definition defines a group of order not greater than $8\beta^2$ in which the order of $t_1 t_2^3$ is 2β. This second group is a quotient group of the first group. That groups defined by the above conditions actually exist for every value of α and β remains to be proved. The two definitions will be considered independently.

* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 60.

Case I. $t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1.$

Consider the generators

$$t_1 = abcd \cdot efgh \cdot ijkl \cdot \cdots \cdot mnop \cdot qrst \cdot uvwx,$$

$$t_2 = cdef \cdot ghij \cdot kl - \cdots - mn \cdot opqr \cdot stuv \cdot wxab,$$

where both t_1 and t_2 contain α cycles of order 4. Then

$$t_1 t_2 = aw \cdot bd \cdot ce \cdot fh \cdot gi \cdot k - \cdots - m \cdot np \cdot oq \cdot rt \cdot su \cdot vx,$$

$$t_1 t_2^3 = bfj - \cdots - nrv \cdot dxtp - \cdots - lh,$$

$$t_1^2 t_2^2 = aei - \cdots - mqu \cdot bfj - \cdots - nrv \cdot cwso - \cdots - kg \cdot dxtp - \cdots - lh.$$

It is evident that the above substitutions satisfy the conditions necessary and sufficient to generate the group of order $4\alpha^2$. By adding another cycle of order 4 to t_1, say $\alpha\beta\gamma\delta$, and changing t_2 to

$$cdef \cdot ghij \cdot kl - \cdots - mn \cdot opqr \cdot stuv \cdot wx\alpha\beta \cdot \gamma\delta ab,$$

it follows immediately by mathematical induction that generators satisfying the required conditions exist for every value of α. When $\alpha = 1$, G is the cyclic group of order 4. For $\alpha = 2$, G is the non-abelian group of order 16 which contains only three squares and has eight operators of order 4 and seven operators of order 2. When $\alpha = 3$, the group generated is of order 36 and contains eighteen operators of order 4, nine operators of order 2 and eight operators of order 3. Hence the existence of this system of groups is established and the following theorem is proved:

THEOREM I. *Two operators t_1 and t_2 which satisfy the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1$$

generate a group whose order is $4\alpha^2$ at the greatest, and such a group exists for every value of α.

The orders of all the operators of G are known when α is known. Since

$$t_1^2 (t_1^3 t_2) t_1^2 = t_1 t_2 t_1^2 = t_2^3 t_1 = (t_1^3 t_2)^{-1},$$

and

$$t_1^2 (t_1 t_2^3) t_1^2 = t_1^3 t_2^3 t_1^2 = t_2 t_1^3 = (t_1 t_2^3)^{-1},$$

then t_1^2 transforms the generators of H into their inverses, respectively, and hence the product of t_1^2 and any operator of H is of order 2, that is, *all the elements of the coset* $t_1^2 H$ *are of order* 2. Now H is the direct product of the cyclic groups generated by $t_1 t_2^3$ and $t_1^3 t_2$ and hence every operator in it is a product of powers of these two generators. Let $s = (t_1 t_2^3)^\delta \cdot (t_1^3 t_2)^\theta$, where $\delta, \theta = 1, 2, \ldots, (\alpha - 1)$. Then

$$(t_1 s)^4 = t_1^2 (t_2^3 t_1)^\delta t_1^2 (t_2 t_1^3)^\theta t_1^2 (t_1 t_2^3)^\delta t_1^2 \cdot t_1^2 (t_1^3 t_2)^\theta t_1^2 (t_2^3 t_1)^\delta t_1^2 (t_2 t_1^3)^\theta t_1^2 (t_1 t_2^3)^\delta (t_1^3 t_2)^\theta$$

$$= (t_2^3 t_1)^{-\delta} (t_2 t_1^3)^\theta (t_1 t_2^3)^{-\delta} (t_1^3 t_2)^{-\theta} \cdot (t_2^3 t_1)^\delta (t_2 t_1^3)^{-\theta} (t_1 t_2^3)^\delta (t_1^3 t_2)^\theta = 1,$$

since $t_1 t_2^3$ and $t_1^3 t_2$ are commutative. From the relation $t_1 s = (s^{-1} t_1^3)^{-1}$ and the above we get $(t_1^3 s_1^{-1})^4 = 1$ and $(t_1^3 s)^4 = 1$ on account of the range of values of δ and θ. Now $t_1 s$ cannot be of order 2, for, if so, the relation $(t_1^2 s)^2 = 1$ reduces to $t_1 s^{-1} t_1 s = 1$, whence s is of order 2, that is, $\alpha = 2$; but when $\alpha = 2$, $s = t_1^2 t_2^2$, and $t_1 s = t_1^3 t_2^2$, which is of order 4 and contradicts the assumption that $t_1 s$ is of order 2. Hence $t_1 s$ and $t_1^3 s$ are both of order 4 and therefore, *all the operators of the cosets* $t_1 H$ *and* $t_1^3 H$ *are of order* 4.

When α is odd, the operators in the coset $t_1^2 H$ are all conjugates, and since t_1^2 is a square of an operator in each of the cosets $t_1 H$ and $t_1^3 H$, then all the operators in the coset $t_1^2 H$ are squares of operators in the cosets $t_1 H$ and $t_1^3 H$. When α is even there are five sets of conjugate operators of order 2 in G. Two of these sets are in H, the first being the invariant operator $(t_1 t_2^3)^{\alpha/2} (t_1^3 t_2)^{\alpha/2}$, and the second set is composed of the remaining two operators of order two, namely, $(t_1 t_2^3)^{\alpha/2}$ and $(t_1^3 t_2)^{\alpha/2}$, which are always in H when α is even. The remaining three sets of conjugates are in the coset $t_1^2 H$, for when α is even the operators $(t_1 t_2^3)^2$ and $(t_1^3 t_2)^2$ generate an abelian group of order $\alpha^2/4$. The quotient group on this as a head is of order 16. Let J be this group of order $\alpha^2/4$. Then

$$H = J + t_1^2 t_2^2 J + t_1 t_2^3 J + t_1^3 t_2 J,$$

and

$$t_1^2 H = t_1^2 J + t_2^2 J + t_1^3 t_2^3 J + t_1 t_2 J.$$

Transforming $t_1^2 H$ by t_1, $t_1^2 J$ and $t_2^2 J$ are left invariant and $t_1^3 t_2^3 J$ and $t_1 t_2 J$ are transformed into each other. Hence there are three sets of conjugate operators of order 2 in $t_1^2 H$ when α is even, and therefore five sets in G.

H is invariant under G, and since $t_2^{-1}\, t_1^2\, t_2 = t_2^3\, t_1^2\, t_2 = t_2^2 \cdot t_2\, t_1^2\, t_2$, of which both t_2^2 and $t_2\, t_1^2\, t_2$ are in the group generated by t_1^2 and H, it follows that H and t_1^2 generate an invariant subgroup of G of order $2\alpha^2$; hence G *is a solvable group.*

The subgroup generated by t_1^2 and H is an extended dihedral group and is represented by $H + t_1^2 H$.

The operators $t_1^2\, t_2^2$, $t_2^2\, t_1^2$, $t_1\, t_2^2\, t_1$ and $t_2\, t_1^2\, t_2$ are commutators by Lemma III. Now

$$t_1\, t_2^2\, t_1 \cdot t_1^2\, t_2^2 = t_1\, t_2^2\, t_1^3\, t_2^2 = t_1\, t_2^2\, t_1^3\, t_2^3\, t_2^3 = t_1\, t_2^3\, t_1\, t_2^3 = (t_1\, t_2^3)^2.$$

Therefore $(t_1\, t_2^3)^2$, and likewise $(t_1^3\, t_2)^2$, is in the commutator subgroup, which will be designated by K. $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ are also commutators. If α is odd, $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ generate $t_1\, t_2^3$ and $t_1^3\, t_2$, respectively, so that the order of K is at least α^2, and hence K is H or includes H as a subgroup. The quotient group on H as a head is of order 4, and hence abelian, and since the commutator quotient group must be the largest abelian group complementary to an invariant subgroup of G*, it follows that *when α is odd K coincides with H.*

Since t_1^2 transforms the operators of H into their inverses, the squares of the generators of H are commutators and generate an invariant subgroup of order $\alpha^2/4$; that is, $(t_1\, t_2^3)^2$ and $(t_1^3\, t_2)^2$ generate an invariant subgroup of order $\alpha^2/4$. The quotient group on this group of order $\alpha^2/4$ as a head is of order 16 and non-abelian, since, if it were abelian, K would be of order not greater than $\alpha^2/4$. But

$$(t_1\, t_2^3)^2 \cdot (t_1^3\, t_2)^2 = (t_1^2\, t_2^2)^2,$$

and when α is even $(t_1^2\, t_2^2)^2$ will not generate $t_1^2\, t_2^2$, which is a commutator. Accordingly, K is of order at least $\alpha^2/2$. The quotient group on this group of order $\alpha^2/4$ as a head is a group of order 16 having an abelian head of type $(1, 1, 1)$ and its remaining operators of order 4. The quotient group on the group of order $\alpha^2/2$ as head is of order 8 and abelian, type $(2, 1)$, since this group of order $\alpha^2/2$ has a quotient group of order 2 with regard to H and the quotient group on H as head with regard to G is the cyclic group of order 4. Hence K *is of order $\alpha^2/2$ when α is even.*

As previously stated, G contains the extended dihedral group $H + t_1^2 H$, and since t_1^2 transforms each of the operators of H into its inverse, t_1^2 and every subgroup of H generate a group which is dihedral when the subgroup

* Miller, Blichfeldt and Dickson, *Finite Groups*, p. 69.

of H is cyclic, and which is an extended dihedral group when the subgroup of H is not cyclic. The number of cyclic subgroups of order α in H is $\alpha+1$, so that t_1^2 and these $\alpha+1$ cyclic subgroups generate $\alpha+1$ dihedral subgroups of order 2α.

When α is odd, G contains no invariant operators except identity. When α is even the operator $(t_1 t_2^3)^{\alpha/2}\cdot(t_1^3 t_2)^{\alpha/2}$ is invariant, for t_1 and t_2 transform $t_1 t_2^3$ into the inverse of $t_1^3 t_2$, and $t_1^3 t_2$ into $t_1 t_2^3$, and t_1^2 and t_2^2 transform $t_1 t_2^3$ and $t_1^3 t_2$ into their inverses, respectively, and t_1^3 and t_2^3 transform $t_1 t_2^3$ into $t_1^3 t_2$, and $t_1^3 t_2$ into the inverse of $t_1 t_2^3$. Since all the operators of H are products of powers of $t_1^3 t_2$ and $t_1 t_2^3$, let $(t_1 t_2^3)^{\delta}(t_1^3 t_2)^{\theta}$ be any operator of H, and let s be any operator of G not in H. Then from the above it follows that s transforms $(t_1 t_2^3)^{\delta}(t_1^3 t_2)^{\theta}$ into one of the following:

$$(t_1^3 t_2)^{-\delta}(t_1 t_2^3)^{\theta}, \qquad (t_1 t_2^3)^{-\delta}(t_1^3 t_2)^{-\theta}, \qquad (t_1^3 t_2)^{\delta}(t_1 t_2^3)^{-\theta}.$$

For $(t_1 t_2^3)^{\delta}(t_1^3 t_2)^{\theta}$ to be invariant under s we must have $\theta \equiv -\delta$, $\delta \equiv \theta$, $\delta \equiv -\delta$, $\theta \equiv -\theta$, or $\delta \equiv -\theta$, $\theta \equiv \delta$, mod α, and these can be true only when $\theta = \delta = \alpha/2$. Hence *the central of G is identity when α is odd, and it is of order 2 when α is even.*

This system of groups is closely associated with the elliptic modular theory, the parameter α being the modulus. In fact they are the groups associated with the elliptic norm curve C_α when the invariant $g_3 = 0$. Consider a system of homogeneous substitutions based on the periods ω_1 and ω_2 such that*

$$S: \omega_1 = \omega_1\,\omega_2, \qquad \omega_2' = \omega_2,$$

$$T: \omega_1' = -\omega_2, \qquad \omega_2' = \omega_1.$$

Then T is of order 4. Let the system be the following:

$$P_1: \pi_1 x_i' = \varepsilon x_i,$$

$$P_2: \pi_2 x_i' = x_{i+1},$$

$$P_3: \pi_3 x_i' = x_{-i} \qquad (i = 0, 1, 2, \ldots, (\alpha-1)),$$

where $\varepsilon = e^{2\pi i/\alpha}$, $i = \sqrt{-1}$. These generate the collineation group $G_{2\alpha^2}$ of the elliptic norm curve C_α†. Then the following relations hold:

* Klein-Fricke, *Theorie der Elliptischen Modulfunktionen*, vol. 1, p. 219.

† Ibid., vol. 2, pp. 264 ff.

$$P_1^\alpha = P_2^\alpha = 1, \qquad P_1 P_2 = P_2 P_1, \qquad P^2 = 1,$$

$$P P_1 P^{-1} = P_1, \quad P P_2 P^{-1} = P_2, \qquad T^2 = P,$$

$$T^{-1} P T = P, \quad T^{-1} P_1 T = P_2, \quad T^{-1} P_2 T = P_1^{-1}.$$

$P_1^i P_2^j$ will be of order α when i and j are relatively prime to α. Since P transforms P_1 and P_2 into their inverses, respectively, P, P_1 and P_2 generate an extended dihedral group, and the product of P and any operator of the abelian group generated by P_1 and P_2 will be of order 2; that is, $P_1^i P_2^j P$ is of order 2. Furthermore, $P_1^i P_2^j T$ is of order 4, since

$$P_1^i P_2^j T \cdot P_1^i P_2^j T = P_1^i P_2^j T P_1^i P_2^j T^{-1} \cdot T^2 = P_1^i P_2^j P_2^{-i} P_1^j T^2 = P_1^{i-j} P_2^{j-i} P,$$

which is of order 2.

Let $t_1 = P_1^i P_2^j T$ and $t_2 = T$. Then $t_1 t_2 = P_1^i P_2^j T^2 = P_1^i P_2^j P$, which is of order 2, and $t_1 t_2^3 = P_1^i P_2^j$, which is of order α. Hence these substitutions fulfil the conditions of the definition under discussion, and, therefore, *the system of groups associated with the elliptic norm curve C_α, when the invariant $g_3 = 0$, is the system defined by the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1.$$

When $\alpha = 3$, the G_{36} of this system is a subgroup of the Hesse G_{216} and the simple G_{360}, two of the three most important groups in the study of the geometry of the plane*.

All the preceding results are summarized in the following theorem:

THEOREM II. *Two operators t_1 and t_2 which fulfil the conditions*

$$t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1 t_2^3)^\alpha = 1$$

define a group of order $4\alpha^2$ which exists for every value of α. It is the group of the elliptic norm curve C_α for the invariant $g_3 = 0$. It is solvable. The commutator subgroup is of order α^2 when α is odd and coincides with the abelian subgroup H generated by $t_1 t_2^3$ and $t_1^3 t_2$, but when α is even it is of order $\alpha^2/2$. The central is identity when α is odd, but it is of order 2 when α is even. The group contains $\alpha + 1$ dihedral subgroups of order 2. All the operators of the

* Klein-Fricke, vol. 2, pp. 250ff.

cosets $t_1 H$ and $t_1^2 H$ are of order 4, and those of the coset $t_1^2 H$ are all of order 2, where $G = H + t_1 H + t_1^2 H + t_1^3 H$.

Case II. $t_1^4 = t_2^4 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\beta = 1$.

As was stated before, this group is one whose generators satisfy the conditions of Case I, and hence many of its properties are analogous to those of the groups of Case I. However, the abelian group H is of order $2\beta^2 = \alpha^2/2$, where $\alpha = 2\beta$ is the order of the operators $t_1 t_2^3$ and $t_1^3 t_2$.

Let $p_1 = \alpha\beta\gamma\delta \cdot \varepsilon\theta\varrho\varphi$, $p_2 = \alpha\theta\gamma\varphi \cdot \beta\varrho\delta\varepsilon$. Then

$$p_1 p_2 = \alpha\varrho \cdot \beta\varphi \cdot \gamma\varepsilon \cdot \delta\theta, \quad p_1 p_2^3 = \alpha\varepsilon \cdot \beta\theta \cdot \gamma\varrho \cdot \delta\varphi, \quad p_1^2 p_2^2 = 1.$$

Therefore $p_1^4 = p_2^4 = (p_1 p_2)^2 = p_1^2 p_2^2 = (p_1 p_2^3)^2 = 1$. Now consider the generators $p_1 t_1$ and $p_2 t_2$, t_1 and t_2 being the generators in Case I. p_1 and p_2 are commutative with t_1 and t_2. Applying the properties belonging to p_1 and p_2 and t_1 and t_2, we have

$$(p_1 t_1)^4 = (p_2 t_2)^4 = (p_1 t_1 p_2 t_2)^2 = [(p_1 t_1)^2 (p_2 t_2)^2]^\beta$$
$$= [(p_1 t_1)(p_2 t_2)^3]^{2\beta} = 1,$$

which is fulfilled when β is odd, that is, a system of groups exists and is defined for all odd values of β.

Now set $r_1 = ABCD \cdot EFGH$, $r_2 = ACEG \cdot BF$. Then

$$r_1 r_2 = AF \cdot BE \cdot CD \cdot GH, \; r_1 r_2^3 = AFEB \cdot CDGH, \; r_1^2 r_2^2 = AG \cdot BD \cdot CE \cdot FH.$$

By the same device used above,

$$(r_1 t_1)^4 = (r_2 t_2)^4 = (r_1 t_1 r_2 t_2)^2 = [(r_1 t_1)^2 (r_2 t_2)^2]^{2\Phi}$$
$$= [(r_1 t_1)(r_2 t_2)^3]^{4\Phi} = 1$$

defines and proves the existence of groups distinct from those above for odd values of φ, where $\alpha = 2\beta = 4\varphi$.

Hence these sets of generators prove the existence of groups for all values of α not divisible by 4. No general generators have yet been found.

III. The infinite system of non-abelian groups defined by $t_1^3 = t_2^6 = (t_1 t_2)^2 = 1$, and one additional condition

From the conditions

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = 1, \qquad t_1 t_2 \neq t_2 t_1,$$

the following table of orders of products of powers of t_1 and t_2 is derived:

1	t_2	t_2^2	t_2^3	t_2^4	t_2^5
t_1	2	3	6	α	6
t_1^2	6	α	6	3	2

The only products whose orders appear to be arbitrary are $t_1^2 t_2^2$ and $t_1 t_2^4$, and their order is set equal to α and the conditions to be imposed upon α will now be determined.

The operators $t_1^2 t_2^2$ and $t_2^2 t_1^2$ are commutative, for

$$(t_1^2 t_2^2)^{-1} t_2^2 t_1^2 t_1^2 t_2^2 = t_2^4 t_1 t_2^2 t_1 t_2^2 = t_2^3 t_1^2 t_1^2 t_2 = t_2^3 t_1 t_2 = t_2^2 t_1^2,$$

and hence they generate an abelian group whose order is α^2 at the most. Call this abelian group H. If H contains t_1, it will contain t_2^2, and t_1 and t_2^2 will be commutative. Likewise, if it contains t_2, it will contain t_1, and t_2 and t_1 will be commutative. H is invariant under t_1 and t_2, for

$$t_1^{-1} t_1^2 t_2^2 t_1 = t_1 t_2^2 t_1 = t_2^4 t_1 t_1 t_2^4 = (t_1^2 t_2^2)^{-1} (t_2^2 t_1^2)^{-1},$$

$$t_1^{-1} t_2^2 t_1^2 t_1 = t_1^2 t_2^2, \qquad t_2^{-1} t_1^2 t_2^2 t_2 = t_2^5 t_1^2 t_2^3 = t_1 t_2^4 = (t_2^2 t_1^2)^{-1},$$

$$t_2^{-1} t_2^2 t_1^2 t_2 = t_2 t_1^2 t_2 = t_1^2 t_2^4 t_1^2 = t_1^2 t_2^2 \cdot t_2^2 t_1^2.$$

In order to obtain the group of greatest possible order we assume that H contains neither t_1 nor t_2. Then t_2 and H generate a group of order $6\alpha^2$ at most. Call this group G. Then G is defined by the conditions

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1.$$

To prove the existence of this system of groups of order $6\alpha^2$, set

$$t_2 = abcdef \cdot ghijkl \cdot mnopqr \cdots\cdots stuvwx,$$

$$t_1 = fhd \cdot lnj \cdot r - p \cdots\cdots - t - \cdot xbv \cdot egi \cdot kmo \cdot q - \cdots\cdots - su \cdot wac,$$

where t_2 is composed of α cycles of order 6. Then

$$t_2\, t_1 = av \cdot bw \cdot cf \cdot dg \cdot eh \cdot il \cdot jm \cdot kn \cdot or \cdot p - \cdot q - \cdots\cdots - s \cdot t - \cdot ux,$$

$$t_2^2\, t_1^2 = cio - \cdots - u \cdot bhn - \cdots - t \cdot ew - \cdots - qk \cdot fx - \cdots - rl,$$

of order α,

$$t_1^2\, t_2^2 = as - \cdots - mg \cdot djp - \cdots - v \cdot ekq - \cdots - w \cdot bt - \cdots - nh,$$

of order α.

The proof that these generators fulfil the conditions is by mathematical induction, as for the system of groups in Section II. For $\alpha = 1$ the group is the cyclic G_6, and for $\alpha = 2$ the group is the non-abelian G_{24} having eight operators of order 6, eight operators of order 3 and seven operators of order 2, or the direct product of the tetrahedral group and a group of order 2.

The orders of all the operators of G are known when α is known. Since

$$t_2^3\,(t_1^2\, t_2^2)\, t_2^3 = t_2^3\, t_1^2\, t_2^5 = t_2^4\, t_1 = (t_1^2\, t_2^2)^{-1},$$

$$t_2^3\,(t_2^2\, t_1^2)\, t_2^3 = t_2^5\, t_1^2\, t_2^3 = t_1\, t_2^4 = (t_2^2\, t_1^2)^{-1},$$

t_2^3 transforms all the operators of H into their inverses, respectively, and since t_2^3 is of order 2, the product of t_2^3 and any operator of H is of order 2, that is, all the operators of the coset $t_2^3 H$ are of order 2. The operators of the cosets $t_2^2 H$ and $t_2^4 H$ are all of order 3, for, since any operator of H is a product of powers of its generators, we have

$$\begin{aligned}[t_2^2\,(t_1^2\, t_2^2)^\delta\,(t_2^2\, t_1^2)^\Phi]^3 &= (t_2^2\, t_1^2)^\delta\; t_2 \cdot t_2^3\,(t_1^2\, t_2^2)^{\Phi+\delta}\,(t_2^2\, t_1^2)^{\Phi+\delta}\; t_2^3 \cdot t_2^{-1}\,(t_2^2\, t_1^2)^\Phi \\ &= (t_2^2\, t_1^2)^\delta\; t_2\,(t_1^2\, t_2^2)^{-(\Phi+\delta)}\,(t_2^2\, t_1^2)^{-(\Phi+\delta)}\; t_2^{-1}\,(t_2^2\, t_1^2)^\Phi \\ &= (t_2^2\, t_1^2)^\delta\,(t_1^2\, t_2^2)^{-(\Phi+\delta)}\,(t_2^2\, t_1^2)^{-(\Phi+\delta)}\,(t_1^2\, t_2^2)^{\Phi+\delta}\,(t_2^2\, t_1^2)^\Phi = 1,\end{aligned}$$

for

$$t_2 (t_1^2 t_2^2) t_2^{-1} = t_2 t_1^2 t_2 = t_2 t_1 t_1 t_2 = t_1^2 t_2^1 t_1^2 = t_1^2 t_2^2 \cdot t_2^2 t_1^2,$$

and

$$t_2 (t_2^2 t_1^2) t_2^{-1} = t_2^3 t_1^2 t_2^5 = t_2^4 t_1 = (t_1^2 t_2^2)^{-1}.$$

Since $t_2^4 = t_2^{-2}$, all the operators of the coset $t_2^4 H$ are of order 3. In like manner one proves that the operators of the cosets $t_2 H$ and $t_2^5 H$ are all of order 6.

On account of the fact that t_2^3 is of order 2 and transforms each of the operators of H into its inverse, t_2^3 and H generate an invariant extended dihedral subgroup of G of order $2\alpha^2$, for t_1 transforms t_2^3 into $t_1^2 t_2^3 t_1 = t_1^2 t_2^2 \cdot t_2 t_1$, and the product $t_1^2 t_2^2 \cdot t_2^3 = t_1^2 t_2^5 = (t_2 t_1)^{-1}$, so that $t_2 t_1$ is in the group generated by t_2^3 and H. *Therefore G is a solvable group.*

No operator of H except identity is invariant under G, for $t_2^{-1} (t_1^2 t_2^2)^\delta (t_2^2 t_1^2)^\varphi t_2 = (t_2^2 t_1^2)^{\varphi-\delta} (t_1^2 t_2^2)^\varphi$, and if $(t_1^2 t_2^2)^\delta (t_2^2 t_1^2)^\varphi$ is to be invariant, then $\varphi - \delta = \varphi$ and $\delta = \varphi$, which equations are satisfied only for $\varphi = \delta = 0$. And no operator of G not in H can be invariant under G, for it and H would generate an abelian group of order greater that of H, which is not possible since G is generated by t_2 and H. Hence *the central of G is identity.*

Now $t_1^2 t_2^2$ and $t_2^2 t_1^2$, which generate H, are commutators by Lemma III, and hence K, the commutator subgroup, either coincides with H or contains H as a subgroup. However, the quotient group on H as a head is the cyclic G_6, which is abelian and hence is the maximal abelian quotient group, since K is of order at least α^2. Therefore *K coincides with H and H is the commutator subgroup.*

Since H contains $\alpha + 1$ cyclic subgroups of order α, and since t_2^3 transforms each operator of H into its, inverse, G contains at least $\alpha + 1$ dihedral subgroups of order 2α. t_2^3 and any subgroup of H generate either a dihedral group or an extended dihedral group according as the subgroup of H is cyclic or not.

The group G is closely associated with the theory of elliptic modular functions, being the group of the elliptic norm curve C_a when the invariant $g_2 = 0$*.

Consider

$$S: x_i' = \varepsilon^{\frac{i(\alpha-i)}{2}} x_i, \qquad T: x_i' = \sum_{j=0}^{\alpha-1} \varepsilon^{ij} x_j \qquad (i = 0, 1, 2, \cdots, (\alpha - 1))\dagger.$$

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* Klein-Fricke, *Theorie der Elliptischen Modulfunktionen*, vol. II, p. 242, footnote.

† Ibid, vol. II, p. 292.

Hence $S^\alpha = 1$, $T = 1$. Now set $U = ST$. Then

$$U : x_i' = \sum_{j=0}^{\alpha-1} \varepsilon^{ij} \varepsilon^{\frac{j(\alpha-j)}{2}} x_j, \quad U^6 = 1.$$

Now consider the set of transformations

$$P_1 : \pi_1 x_i' = \varepsilon^i x_i, \qquad P_2 : \pi_2 x_i' = x_{i+1}, \qquad P : \pi_3 x_i' = x_{-i}.$$

Then

$$SP_1 = P_1 S, \quad SP_2 = P_1^{-1} P_2 S, \quad PS = SP, \quad P_1 P_2 = P_2 P_1, \quad PP_1 = P_1^{-1} P,$$

$$PP_2 = P_2^{-1} P, \quad T^{-1} P_1 T = P_2, \quad T^{-1} P_2 T = P_1^{-1},$$

$$UP_1 = STP_1 = SP_2^{-1} T = P_1 P_2 ST = P_1 P_2 U,$$

$$UP_2 = STP_2 = SP_1 T = P_1 ST = P_1 U, \quad U^3 = P.$$

As in Section II, $P_1^i P_2^j$ will be of order α when either i or j is relatively prime to α. Likewise $P_1^i P_2^j P$ will be of order 2, and the product of U and any operator in the abelian group generated by P_1 and P_2 will be of order 6.

Now set $t_1 = U^2$ and $t_2 = P_1^i P_2^j U$. Then $t_2 t_1 = P_1 P_2 U^3 = P_1 P_2 P$, which is of order 2, and

$$t_2^2 t_1^2 = P_1^i P_2^j U \cdot P_1^i P_2^j U \cdot U^4 = P_1^i P_2^j P_1^i P_2^i P_1^j U^6 = P_1^{2i+j} P_2^{i+j},$$

which will be of order α provided either $2i+j$ or $i+j$ is prime to α. Such an operator always exists, since α cannot contain both $2i+j$ and $i+j$ for all values of i and j from zero to α. As a special case, $i = 0$, $j = 1$ gives $P_1 P_2$ which is always of order α. Therefore the conditions $t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1$ are fulfilled and G is the group of the elliptic norm curve C_α for the invariant $g_2 = 0$.

All the preceding results are summarized in the following:

THEOREM III. *If two operators t_1 and t_2 fulfil the conditions*

$$t_1^3 = t_2^6 = (t_1 t_2)^2 = (t_1^2 t_2^2)^\alpha = 1,$$

they generate a group of order $6\alpha^2$. Such a group exists for all values of α. It is the group of the elliptic norm curve C_α for the invariant $g_2 = 0$. It is

solvable and has identity for its central. The commutator subgroup is of order α^2 *and is abelian, being coincident with the abelian head* H *which is generated by* $t_1^2\, t_2^2$ *and* $t_2^2\, t_1^2$. *The operators of the cosets* $t_2 H$ *and* $t_2^5 H$ *are all of order* 6, *those of the cosets* $t_2^3 H$ *and* $t_2^4 H$ *are all of order* 3, *and those of the coset* $t_2^3 H$ *are all of order* 2. *The group contains* $\alpha + 1$ *cyclic subgroups of order* α *and* $\alpha + 1$ *dihedral groups of order* 2.

IV. THE INFINITE SYSTEM OF NON-ABELIAN GROUPS DEFINED BY $t_1^3 = t_2^3 = (t_1\, t_2)^3 = 1$, AND ONE ADDITIONAL CONDITION

If one sets up the table of orders of products it will be found that the orders of the operators $t_1^2\, t_2$ and $t_1\, t_2^2$ are arbitrary.

The operators $t_1^2\, t_2$ and $t_2\, t_1^2$ are commutative, for $t_1^2\, t_2 \cdot t_2\, t_1^2 = t_1^2\, t_2^2\, t_1^2 = t_2\, t_1\, t_2 = t_2\, t_1^2 \cdot t_1^2\, t_2$, and hence they generate an abelian group of order α^2 at most. Call this abelian group H. If H contains either t_1 or t_2, it contains both, contrary to the hypothesis that t_1 and t_2 are not commutative. H is invariant under both t_1 and t_2, for these operators transform the generators of H either into each other or into their product or the product of their inverses. Accordingly, t_1 and H generate a group of order $3\alpha^2$ at the greatest, which is defined by the conditions

$$t_1^3 = t_2^3 = (t_1\, t_2)^3 = (t_1^2\, t_2)^\alpha = 1,$$

where α can have any integral value, as will now be shown. Set

$$t_1 = abc \cdot def \cdot \cdots\cdots ghi \cdot jkl,$$

containing α cycles of order 3, and set

$$t_2 = bcd \cdot ef - \cdots\cdots - g \cdot hij \cdot kla,$$

from which it follows readily that $t_1\, t_2$ is of order 3 and that $t_1^2\, t_2$ and $t_2\, t_1^2$ are of order α and commutative. Hence these substitutions fulfil the conditions of the above definition and prove the existence of the group for all values of α. The generality of the above substitutions is proved by mathematical induction as in the preceding sections. When $\alpha = 1$, the group is the cyclic G_3 and when $\alpha = 3$ the group is the tetrahedral group.

Let G represent the group of order $3\alpha^2$. Then $G = H + t_1 H + t_1^2 H$. The operators of the cosets $t_1 H$ and $t_1^2 H$ are all of order 3, for all the operators in H are products of powers of $t_1 t_2^2$ and $t_2^2 t_1$, and

$$|t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi|^3 = t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi t_1^2 \cdot t_1^2 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi t_1 (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi$$

$$= (t_1^2 t_2)^\delta (t_2 t_1^2)^\delta (t_1 t_2^2)^\Phi (t_2^2 t_1)^\delta (t_1^2 t_2)^\Phi (t_2 t_1^2)^\Phi (t_1 t_2^2)^\delta (t_2^2 t_1)^\Phi$$

$$= 1.$$

Since H is invariant under G and abelian, the group G is solvable.

$t_1^{-1} t_2 t_1 t_2^{-1} = t_1^2 t_2 \cdot t_1 t_2^2$ is a commutator, and it is of order α, since it is the product of two commutative generators of H which are both of order α. $t_1^2 t_2^2 t_1 t_2 = t_1^2 t_2 \cdot t_2 t_1 t_2 = t_1^2 t_2 \cdot t_1^2 t_2^2 t_1^2 = (t_1^2 t_2) \cdot t_2 t_1^2$ is also a commutator. These two commutators generate H when α is not divisible by 3, for, upon setting $r = t_1^2 t_2$ and $s = t_1 t_2^2$, the two commutators are rs and $r^2 s^{-1}$, and making use of the fact that r and s are commutative, it follows that $rs \cdot r^2 s^{-1} = r^3$, which generates r when α is not divisible by 3. Hence when α is not divisible by 3 the commutator subgroup coincides with H, since the quotient group on H as a head is the cyclic G_3. When α is divisible by 3, G contains an invariant subgroup of order $\alpha^2/3$ generated by r^3 and rs, for

$$t_1^{-1} r t_1 = t_1^{-1} t_1^2 t_2 t_1 = t_1 t_2^2 \cdot t_2^2 t_1 = sr^{-1}, \quad t_1^{-1} s t = t_2^2 t_1 = r^{-1},$$
$$(1)$$
$$t_2^{-1} r t_2 = t_2^2 t_1^2 t_2^2 = t_2^2 t_1 \cdot t_1 t_2^2 = sr^{-1}, \text{ and } t_2^{-1} s t_2 = t_2^2 t_1 = r^{-1}.$$

The quotient group on this group as a head is of order 9 and hence abelian, and therefore this group of order $\alpha^2/3$ is the commutator subgroup K. Hence the commutator subgroup K coincides with H except when α is divisible by 3 and then it is of order $\alpha^2/3$.

Now $t_1^{-1} rs t_1 = r^{-2} s$, $t_2^{-1} rs t_2 = r^{-2} s$, and hence both t_1 and t_2 transform $(rs)^\delta$ into $r^{-2\delta} s^\delta$, so that when α is divisible by 3 the operator $(rs)^{\alpha/3}$ is invariant under G. On account of the relations (1) there are no other invariant operators in G except $(rs)^{\alpha/3}$ and its powers when α is divisible by 3, and G has no invariant operators except identity when α is not divisible by 3, for r and s generate H. Therefore the central of G is identity except when α is divisible by 3 and then it is of order 3 and is generated by $r^{\alpha/3} s^{\alpha/3}$. This fact is also easily seen from a study of the generators r and s, for they are each composed of two cycles of order α and they have one common cycle. Hence rs

is composed of two cycles of order α and the square of the third cycle. Raising rs to the power $\alpha/3$ we get a product in which the cycles are all of order 3, and hence this product and its powers offer the only possibilities of invariant operators under t_1 and t_2.

The results of this section are summarized in the following theorem:

THEOREM IV. *Two operators t_1 and t_2 which fulfil the conditions*

$$t_1^3 = t_2^3 = (t_1\, t_2)^3 = (t_1^2\, t_2)^\alpha = 1$$

generate a group of order 3 and such a group exists for every value of α. This group has an abelian head $\widehat{H}$ of order α^2 generated by the operators $t_1^2 t_2$ and $t_2 t_1^2$. The operators of its cosets $t_1 H$ and $t_1^2 H$ are all of order 3. It is a solvable group. When α is not divisible by 3 the central is identity and the commutator subgroup coincides with H. When α is divisible by 3 the central is of order 3 and the commutator subgroup is of order $\alpha^2/3$.

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ON THE INTEGRALS OF ELEMENTARY FUNCTIONS*

BY

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I. INTRODUCTION

This paper contains an extension of Liouville's work on the determination of the circumstances under which the integral of an elementary function is itself elementary.†

The elementary functions are understood here to be those which are obtained by performing algebraic operations and taking exponentials and logarithms. For instance, the function

$$\tan\left[e^{z^2}-\log_z(1+\sqrt{z})\right]+(z^z+\log\arcsin z)^{\frac{1}{9}}$$

is elementary.

Liouville's first result in this field was the theorem that if the integral of an algebraic function of z is elementary, the integral is of the form

$$\alpha_0(z)+c_1\log\alpha_1(z)+c_2\log\alpha_2(z)+\cdots+c_n\log\alpha_n(z),$$

where each $\alpha(z)$ is an algebraic function, and each c a constant.‡ It follows from this theorem that no abelian integral of the first kind is elementary. Liouville also obtained a very broad generalization of the above result, on the basis of which it can be proved, for instance, that the integrals

$$\int e^{z^2}dz, \qquad \int\frac{dz}{\log z},$$

are not elementary functions.§

* Presented to the Society, Feb. 25, 1922.

† No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Journal de l'École polytechnique, vol. 14 (1833), p. 36.

§ Journal für die reine und angewandte Mathematik, vol. 13 (1833), p. 93.

We shall investigate here the circumstances under which the integral of an elementary function can satisfy an elementary transcendental equation. This problem is mentioned by Liouville* and by Hardy,† but nothing seems to have been done towards its solution. Our result is the

THEOREM. *If w, the integral of an elementary function of z, satisfies an equation $F(w, z) = 0$, where $F(w, z)$ is an elementary function of w and z, not identically zero, then w is an elementary function of z.*

Thus no abelian integral of the first kind can satisfy an elementary transcendental equation; nor can, to give special cases again, the integrals of e^{e^z} and of $1/\log z$.

If a function satisfies an elementary transcendental equation, its inverse also satisfies such an equation. It follows from the above that no elliptic function can satisfy an elementary transcendental equation. In particular, no elliptic function is elementary.‡

Whereas, notwithstanding their formal character, little need be added to Liouville's papers to make them rigorous, the work in the present paper has to be largely function-theoretic. This is due principally to the fact, seen in § III, that we may not suppose ourselves to be working in a region in which $F(w, z)$ is analytic. On this account, we have had to make, in § II, a careful examination of the elementary functions.

It is natural to inquire as to whether an integral w of an elementary function is itself elementary if it is one of n functions, $w, w_1, \ldots, w_{n-1}$, which satisfy a system of elementary equations

$$F_i(w, w_1, \ldots, w_{n-1}; z) = 0 \qquad (i = 1, 2, \ldots, n).$$

While the formal elements of our proof, given in § IV, can be extended to settle this question affirmatively, we see no way of avoiding certain function-theoretic assumptions, which, though light in almost any other case, would be out of place in a problem of this kind. We shall therefore not write now on this question.

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* Journal de l'École polytechnique, vol. 14 (1833), p. 40.

† *The Integration of Functions of a Single Variable*, Cambridge, 1916, p. 41. Appendix I of this tract contains further references to Liouville's work on elementary functions.

‡ This does not follow, of course, from the fact that the integrals of the first kind are not elementary. I do not know where it is proved in the literature that the elliptic functions are not elementary, but this result of the present paper can be established by the same method which Liouville uses for the integrals. It can be shown that if the inverse of an abelian integral is elementary, it is either algebraic, or of the form e^{az+b}, where a and b are constants.

II. THE ELEMENTARY FUNCTIONS

We shall deal with certain analytic functions of w and z, which we shall call *elementary functions.*

An analytic function of w and z will be said to be *analytic almost everywhere,* if, given any element of the function $P(w-w_0,\ z-z_0)$,* any curve

$$w = \varphi(\lambda), \qquad z = \psi(\lambda) \qquad (0 \leq \lambda \leq 1),$$

where $\varphi(0) = w_0$ and $\psi(0) = z_0$, and given, finally, any positive ϵ, there exists a curve

$$w = \varphi_1(\lambda), \qquad z = \psi_1(\lambda) \qquad (0 \leq \lambda \leq 1), \tag{1}$$

where $\varphi_1(0) = w_0$ and $\psi_1(0) = z_0$, such that

$$|\varphi_1(\lambda)-\varphi(\lambda)| < \epsilon, \qquad |\psi_1(\lambda)-\psi(\lambda)| < \epsilon,$$

for $0 \leq \lambda \leq 1$, and such that the element $P(w-w_0, z-z_0)$ can be continued along the entire curve (1). Roughly speaking, an element of the function, if it cannot be continued along a given path, can be continued along some path in any neighborhood of the given one.

An algebraic function u, given by an irreducible equation

$$\alpha_0(w,z)\,u^m + \alpha_1(w,z)\,u^{m-1} + \cdots + \alpha_m(w,z) = 0, \tag{2}$$

where each α is a polynomial in w and z with constant coefficients, is analytic almost everywhere, for the space of w and z is four-dimensional, and the singularities of u lie along the two-dimensional manifolds obtained by equating $\alpha_0(w,z)$ and the discriminant of (2) to zero†.

In what follows, the algebraic functions will frequently be called *functions of order zero,* and the variables w and z *monomials of order zero.*

The functions e^v and $\log v$, where v is any non-constant algebraic function, are called by Liouville *monomials of the first order.* It is seen directly that e^v is analytic almost everywhere. If v is analytic, and nowhere zero, along a given curve, $\log v$ is analytic along the curve. If v should vanish for some points of the curve, there is a curve arbitrarily close to the given one on which v is everywhere different from zero‡. Thus $\log v$ is analytic almost everywhere.

* It is to be recalled that an analytic element, $P(w-w_0, z-z_0)$, is a series of positive integral powers of $w-w_0$ and $z-z_0$. The point (w_0, z_0) is called the center of the element.

† It is not hard to make the proof formal.

‡ A formal proof can be based upon the fact that in continuing v along the given curve, only a finite number of elements of v need be used.

More generally, we shall say, following Liouville, that u is a *function of the first order,* if it is not algebraic, and if it satisfies an equation like (2), in which each α is a rational integral combination of monomials of orders zero and one.

To obtain each α one must be given, for some point (w_0, z_0), an element of each of the monomials on which that α is based. A proper combination of these elements furnishes an element of the α, from which all other elements of the α are found by continuation. Each α is analytic almost everywhere.

Similarly, in stating that u is determined by (2), we mean that for an element of each of the functions u and α_i $(i = 0, 1, \ldots, m)$, the first member of (2) vanishes. If the first member of (2) is reducible in the domain of rationality of the given elements of the α's, we may replace it by that one of its irreducible factors which vanishes for the given element of u. It may thus be assumed that the discriminant of (2), which is analytic wherever every α is analytic and α_0 is not zero, does not vanish identically. We see now that u is analytic almost everywhere, since in the neighborhood of every curve there is a curve along which each α is analytic, and on which α_0 and the discriminant of (2) are everywhere different from zero.

The functions of orders zero and one form together a set which is closed with respect to all algebraic operations. That is, a function determined by an equation like (2), in which each α is a rational integral combination of functions of orders zero and one, is, itself, either algebraic, or a function of the first order. This fact, which will be very important for us, can be proved in exactly the same way as it is shown that algebraic operations performed upon algebraic numbers lead always to algebraic numbers.*

An exponential or a logarithm of a function of order $n - 1$ is called a *monomial of order n,* provided that it is not among the functions of orders $0, 1, \ldots, n - 1$. With the same reservation, any function defined by an equation like (2), in which each α is a rational integral combination of monomials of orders $0, 1, \ldots, n$, is a *function of order n.* As above, we may assume that the discriminant of (2) does not vanish identically.

One sees by a quick induction that a function of any order n is analytic almost everywhere. For every n, the functions of orders $0, 1, \ldots, n$ form a set which is closed with respect to all algebraic operations.

The functions to which orders are assigned by the preceding definitions will be called *elementary functions* of w and z.

We now inquire as to whether an elementary function of w and z becomes, when w is held fast, an elementary function of z; that is, a function which can

* See, for instance, Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen,* Leipzig, 1918, p. 7.

be obtained from z alone, without introducing a second variable, by performing algebraic operations and taking exponentials and logarithms. Perhaps an illustration is necessary to show that there is a real question here. Consider the equation

$$e^{\cot wz}\, u - z\, e^{\csc wz} = 0.$$

It determines u as an entire function of w and z, which reduces to z when $w = 0$. The equation itself is meaningless when $w = 0$. Certainly, then, it is possible for $u(w, z)$ to be an elementary function of w and z and for $u(w_1, z)$ to be analytic in z, but for the operations which produce $u(w, z)$ to be meaningless when $w = w_1$. In the above example, it is easy to show that $u(w_1, z)$ is an elementary function of z, but we cannot be certain that this is possible in all cases.

We shall not settle this question completely, but shall limit ourselves to deriving a result which will suffice for our later purposes. We deal with an elementary function $u(w, z)$, of which some branch is analytic at a point (w_1, z_1). If w is assigned the fixed value w_1, and if z is allowed to range over the neighborhood of z_1, the given branch of u becomes an analytic function of z. It is the monogenic analytic function of z obtained by continuing this branch in which we are interested.

It is of course obvious that if u is an algebraic function of w and z, u becomes an algebraic function of z if w is held fast. Let v be algebraic in w and z. If $\log v$ is analytic at (w_1, z_1), then v must be analytic at (w_1, z_1). Hence $\log v(w_1, z)$ is an elementary function of z. Consider now e^v. If e^v is analytic at (w_1, z_1), there is a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which e^v is not zero, so that v is analytic at (w_2, z_2).* Hence $e^{v(w_2, z)}$ is an elementary function of z. More generally, if w is held fast at a value sufficiently close to w_2, e^v becomes an elementary function of z.

Suppose now that α is a rational integral combination of monomials of orders zero and one, and that α is analytic at (w_1, z_1). We may suppose that the element of α with center at (w_1, z_1) is obtained by continuing, along a path C, the element of α with center at (w_0, z_0) which is found by multiplications and additions from the given elements of the monomials at (w_0, z_0). As each monomial is analytic almost everywhere, we can continue α from (w_0, z_0) along a path C', very close to C, on which every monomial is analytic. We can reach in this way a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which α has an element which is an immediate continuation of the above mentioned element at (w_1, z_1). Also, from what we saw above, there is a point (w_3, z_3),

* Whether e^v can ever be zero, even at a singularity of v, is a matter with which we need not concern ourselves here.

arbitrarily close to (w_2, z_2), such that if w is held fast at any value sufficiently close to w_2, each monomial becomes an elementary function of z. While it might appear at first that (w_3, z_3) is different for different monomials, it becomes plain quickly that a common point may be used for all. For any value of w close to w_3, α is an elementary function of z.

Suppose now that u, analytic at (w_1, z_1), is given by (2), where each α involves monomials of orders zero and one. There exists a point (w_2, z_2), arbitrarily close to (w_1, z_1), at which each α is analytic, and such that if w is given a fixed value sufficiently close to w_2, each α is an elementary function of z. We may suppose also that (w_2, z_2) is so chosen that the coefficients α_i do not all vanish at (w_2, z_2). In that case, if w is held fast at a value close to w_2, u becomes an elementary function of z.

In the general case, it is clear that if u is an elementary function of w and z, analytic at (w_1, z_1), then there is a point (w_2, z_2), arbitrarily close to (w_1, z_1), such that if w is held fast at any value sufficiently close to w_2, u becomes an elementary function of z.

We consider now the differentiation of the elementary functions. Let u be a function of the first order, defined by an equation like (2). The first member of (2) being irreducible, its partial derivative with respect to u cannot vanish for every w and z. Thus, the partial derivative of u with respect to w has an expression rational in u, the α's, and the derivatives of the α's. The derivative of every α is a rational integral combination of the monomials which enter into that α, and of algebraic functions. It follows that u satisfies an equation like (2), in which no monomials of order one appear which do not appear in the equation for u. Proceeding by induction, we can prove that if u is elementary, u_w and u_z are elementary. We prove in the same induction that if, in the equation (2) which defines u, the highest order of the monomials which involve w* is r, and if there are s distinct such monomials of order r, the derivatives of u satisfy equations like (2), in which the monomials which involve w are at most of order r, and if of that order, are among the s monomials mentioned above. This fact will be used in § IV.

III. Formulation of the problem

When, in what follows, we say that $F(w, z)$ assumes the value zero at (w_0, z_0), we shall mean that there exists an analytic element of $F(w, z)$, with center at (w_0, z_0), which assumes there the value zero. There may be other

* In saying that a monomial involves w, we mean not only that w appears in the operations which produce the monomial, but also that the value of the monomial varies when w varies. Furthermore, from what goes before, we know that if a monomial is independent of w, it can be produced by operations which involve only z.

elements with centers at (w_0, z_0) which are not zero there. When we say that $F(w, z)$ has a singularity at (w_0, z_0), we shall mean that there exists a sequence of elements of $F(w, z)$, each an immediate continuation of the preceding one, whose centers approach (w_0, z_0), and the radii of whose restricted domains of convergence approach zero.* $F(w, z)$ may have a singularity at (w_0, z_0), and also have branches which are analytic at (w_0, z_0).

Consider any elementary function of z alone. Let w represent an integral of this function. Our problem is to determine the circumstances under which there exists an equation

$$F(w, z) = 0, \tag{3}$$

where $F(w, z)$ is an elementary function.

Consider any circle in the plane of z, within and on the boundary of which some branch of w is analytic. Pairing the values of z in this circle with the corresponding values of the mentioned analytic branch of w gives a two-dimensional manifold — call it C — in the space of w and z. It will suffice to assume that (3) holds on such a manifold.

It would be unreasonable to require that $F(w, z)$ be analytic on C. It is even possible that every point of C should be a singular point of $F(w, z)$ and that (3) should serve well to determine w. For instance, the equation $(w - z)^{\frac{1}{2}} = 0$ gives $w = z$, but wherever $w = z$, $(w - z)^{\frac{1}{2}}$ has a singularity. In this particular example, the singularities can be removed by squaring, but we have no assurance at present that there do not exist complicated equations which cannot be freed from singularities.

The best way to meet this situation is to extend our problem. We shall undertake *to determine the circumstances under which, for each point of C, $F(w, z)$ either assumes the value zero or else has a singularity.*† *Our result is the*

THEOREM. *If, at each point of an uncountable set of points of C, $F(w, z)$ either assumes the value zero or has a singularity, w is an elementary function of z.*‡

* The radius of the restricted domain of convergence of an element $P(w - w_0, z - z_0)$ is a number ρ such that the element converges when $|w - w_0| < \rho$, and $|z - z_0| < \rho$, and diverges when $|w - w_0| > \rho$ and $|z - z_0| > \rho$.

† It is understood in this that some of the points of C may be zeros, and the remaining ones singularities.

‡ In using any uncountable set on C in the hypothesis, rather than all of C, we are not seeking generality. Such an uncountable set has to be used in the proof, and it saves time to introduce it immediately.

We need not seek to formulate a notion of $F(w, z)$ assuming the value zero at a singular point. The mere existence of the singular points will permit us to prove that w is elementary.

Let n be the smallest integer such that a function $F(w, z)$ of order n exists which satisfies the hypothesis of our theorem. In what follows, $F(w, z)$ will be assumed to be of order n.

Let us see what happens if $F(w, z)$ is zero for each point of an uncountable set of points of C.

For each such point, $F(w, z)$ has an element with center at the point, which equals zero at its center. There must exist a positive ϵ such that, for some uncountable subset E of those points, the radii of the restricted domains of convergence of the elements of $F(w, z)$ exceed ϵ. If no such ϵ existed, the elements, at the centers of which $F(w, z)$ vanishes, could be denumerated.

Let (a, b) be any point of condensation* of E. Consider the neighborhood of (a, b) given by

$$|w-a| < \frac{\epsilon}{2}, \qquad |z-b| < \frac{\epsilon}{2},$$

and the uncountable subset E' of E which lies in this neighborhood.

We now form for (a, b) the immediate continuations of the elements of $F(w, z)$ with centers at the points of E'. The radius of the restricted domain of convergence of each of these elements exceeds $\epsilon/2$, and each new element equals zero at the center of the element of which it is the immediate continuation.

An infinite number of these immediate continuations must be identical, else there would be an uncountable set of distinct elements of $F(w, z)$ with centers at (a, b), and we would have a contradiction of the well known Poincaré-Volterra theorem.

It follows that one of the elements is zero for an infinite number of points of E'. Such an element is a uniform function of w and z in a four-dimensional region containing part of C. Substituting for w, in this element, its value in terms of z, we have an analytic function of z which vanishes at an infinite number of points of the circle mentioned above, and which is therefore identically zero.

We conclude that $F(w, z)$ is analytic in a four-dimensional region which contains part of C, and vanishes on that part of C.

We consider now the case in which $F(w, z)$ has a singularity at each point of an uncountable set of points of C. Let $u = F(w, z)$ be defined by an

* A point in every neighborhood of which there is an uncountable subset of E.

equation like (2), in which each α is a rational integral combination of monomials of orders not exceeding n. If $F(w, z)$ has a singularity at (a, b), there exists a curve ending at (a, b), along which, except at (a, b), $F(w, z)$ is analytic.* If this curve is subjected to a slight deformation, its end being held fast at (a, b), $F(w, z)$ will also be analytic along the new curve, except at (a, b). As each α in (2) is analytic almost everywhere, we can so choose the new curve that every α is analytic along it, except, perhaps, at (a, b).† If, as the new curve is followed to (a, b), each α should prove to be analytic at (a, b), and if neither α_0 nor the discriminant of (2) should vanish at (a, b), the new curve could not lead to a singularity of $F(w, z)$.

Suppose that there is an α which has an uncountable set of singularities on C. In that case one of the monomials on which that α is based must have an uncountable set of singularities on C. If the monomial is of the form e^v, v must have a singularity wherever the monomial does. If the monomial is of the form $\log v$, v must either assume the value zero, or have a singularity, wherever $\log v$ has a singularity. We find, in any case, a contradiction of the fact that no function of order less than n has an uncountable set of singularities on C, or is zero on such an uncountable set.

Thus, either $\alpha_0(w, z)$ or the discriminant of (2) must be zero on an uncountable set of points of C.

The hypothesis of our theorem has implied the existence of a function of order n, either $F(w, z)$, $\alpha_0(w, z)$, or the discriminant, which is analytic and equal to zero in a neighborhood on C. Let C_1 denote that neighborhood, and let the function — call it u — be defined by an equation like (2).‡ As none of the monomials which appear in the equation (2) for the function can have an uncountable set of singularities on C, there must be a neighborhood C_2, in C_1, for which all of the monomials have analytic branches, which, together with u, satisfy (2).§

As u vanishes in C_2, $\alpha_m(w, z)$, the term independent of u in the equation for u, which, since (2) is irreducible, cannot vanish identically, must vanish in C_2.

IV. Completion of the proof

There is a set — call it A — of rational integral combinations of monomials, each combination of the minimum order n used in § III, which vanish in a

* This statement is equivalent to the definition above of a singular point.

† It should perhaps be emphasized that (a, b) does not correspond to the point (w_0, z_0) used in defining "analytic almost everywhere". It is a point to which we approach by analytic continuation, not from which we start.

‡ If the function is a_0 or the discriminant, the equation will be linear.

§ This statement is easily proved, using the fact that the monomials are analytic almost everywhere.

neighborhood on C, but do not vanish identically;* we have just shown the existence of one such expression. With each expression in A may be associated two numbers; first, r, the maximum of the orders of those monomials in the expression which involve w; second, s, the number of such monomials of order r. Many different expressions will represent the same function; the two numbers will vary with the expression.

Of all the expressions in A, there is a set A_1 for which r has a minimum value r_0. Of the expressions in A_1, there is a set A_2, for which s has a minimum value s_0.

Let u be a particular expression in A_2. Those expressions in A_2 which contain no monomials which do not appear in u constitute a set A_3. All expressions in A_3 contain the same s_0 monomials of order r_0 which involve w. Let θ be one of those monomials.

There is an expression in A_3,

$$\overline{u} = \beta_0 \theta^i + \beta_1 \theta^{i-1} + \cdots + \beta_i,$$

where each β is a polynomial in monomials other than θ, which is of a minimum degree i in θ.

In the neighborhood on C in which $\overline{u}$ vanishes, there is a neighborhood C_1 in which every monomial in the expression for $\overline{u}$ is analytic. There is, in fact, a neighborhood C_1, in which, for every monomial, whether of the form e^v or $\log v$, the function v is analytic.†

The partial derivative

$$\overline{u}_\theta = i\beta_0 \theta^{i-1} + (i-1)\beta_1 \theta^{i-2} + \cdots + \beta_{i-1}$$

cannot vanish for every w and z. If it did, the function $i\overline{u} - \theta\overline{u}_\theta$, which could not vanish identically, since $\overline{u}$ does not, would be in A_3 and would be of degree less than i in θ. Hence $\overline{u}_\theta$ cannot be zero throughout C_1, else it would be a function in A_3 of degree less than i in θ.

Thus there is a neighborhood C_2 in C_1 in which the equation $\overline{u} = 0$ can be turned into the form

$$\theta = f(w, z), \tag{4}$$

where $f(w, z)$ is analytic throughout C_2.

If $r_0 = 0$, (4) states that w is an elementary function of z.

* The neigborhood may be different for different expressions.

† This follows from the fact that no v can have an uncountable set of singularities on C.

Suppose that $\theta = e^{v(w,z)}$, where $v(w, z)$, of order $r_0 - 1$, is analytic throughout C_1. Let (w_0, z_0) be any point of C_2. Equation (4) shows that $f(w_0, z_0)$ is not zero. Let, then, ϱ be a number such that $v(w, z)$ and $\log f(w, z)$ are analytic for $|w - w_0| < \varrho$ and $|z - z_0| < \varrho$. We have, for a neighborhood C_3 in C_2, which contains (w_0, z_0) and lies within the region just described,

$$v(w, z) = \log f(w, z). \tag{5}$$

Let $\varphi(z)$ be the elementary function of which w is the integral. Differentiating in (5) with respect to z, we have, throughout C_3,

$$v_w(w, z)\,\varphi(z) + v_z(w, z) = \frac{f_w(w,z)\,\varphi(z) + f_z(w, z)}{f(w, z)}. \tag{6}$$

Having regard to the fact that the equation (2) for $f(w, z)$ is the equation $\bar{u} = 0$ with θ replaced by u, and to the remarks in the last paragraph of § II, we see that the second member of (6) satisfies an equation (2) in which no monomial which involves w is of order greater than r_0, and in which there are at most $s_0 - 1$ such monomials of order r_0. The first member of (6) satisfies an equation (2) in which every monomial involving w is of order less than r_0.

The two members of (6), considered as functions of w and z, must be identical. If they were not, the integral w would satisfy an elementary equation (6), from which we could derive an expression in A_2 with $r \leqq r_0$, and with $s < s_0$ if $r = r_0$.

The two functions

$$v(w+\mu, z), \qquad \log f(w+\mu, z) \tag{7}$$

are analytic in w, μ and z for $|w+\mu-w_0| < \varrho$, $|z-z_0| < \varrho$. Let δ be any positive number less than ϱ, and let C_4 be any neighborhood in C_3 for which $|w-w_0| < \varrho - \delta$, $|z-z_0| < \varrho$. If μ is assigned a fixed value, less in modulus than δ, and if (w, z) is kept in C_4, the functions (7) become functions of z whose derivatives are given by the members of (6), with $w+\mu$ substituted for w. Since (6), being an identity, will hold after this substitution, the difference of the functions in (7) must stay constant as (w, z) varies over C_4. We have thus, on C_4,

$$v(w+\mu, z) = \beta(\mu) + \log f(w+\mu, z), \tag{8}$$

where $\beta(\mu)$, being the difference of two analytic functions of μ, is analytic for $|\mu| < \delta$. Differentiating in (8) with respect to μ, and putting $\mu = 0$ in the result, we have, on C_4,

$$v_w(w, z) = \beta'(0) + \frac{f_w(w, z)}{f(w, z)}. \tag{9}$$

As above, (9) must be an identity, so that, integrating, we have for $|w - w_0| < \varrho$, $|z - z_0| < \varrho$,

$$v(w, z) = \beta'(0)\, w + \log f(w, z) + \gamma(z), \tag{10}$$

where $\gamma(z)$, analytic for $|z - z_0| < \varrho$, is still to be determined.

As (5) is not an identity, there is a point (w_1, z_1), with $|w_1 - w_0| < \varrho$, $|z - z_0| < \varrho$, which does not satisfy (5). By § II, (w_1, z_1) may be so chosen that $f(w_1, z)$ is an elementary function of z alone. Substituting w_1 for w in (10), and comparing the resulting equation with (10), we obtain, for $|w - w_0| < \varrho$, $|z - z_0| < \varrho$,

$$v(w, z) - \beta'(0)\, w - \log f(w, z) = v(w_1, z) - \beta'(0)\, w_1 - \log f(w_1, z). \tag{11}$$

Because of (5), we must have, in C_5,

$$\beta'(0)\, w = \beta'(0)\, w_1 + \log f(w_1, z) - v(w_1, z). \tag{12}$$

Now $\beta'(0)$ cannot be zero. If it were, $\log f(w_1, z) - v(w_1, z)$ would vanish in an area in the z-plane, and therefore for every z, whereas, by the choice of (w_1, z_1), it cannot vanish at z_1.

Thus (12) gives w as an elementary function of z.

Consider now the case of $\theta = \log v(w, z)$. Equation (4) becomes

$$f(w, z) = \log v(w, z),$$

and with almost no modification in the treatment of (5) other than an interchange of the letters f and v, we find again that w is an elementary function of z.

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INVARIANTS OF THE LINEAR GROUP MODULO p^k*

BY

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INTRODUCTION

Professor L. E. Dickson in his paper *A fundamental system of invariants of the general modular linear group with a solution of the form problem*† gave a fundamental system of invariants for the group of all linear homogeneous transformations in n variables with coefficients in the Galois field of order p^n, denoted by $GF[p^n]$.

Dr. J. S. Turner extended the results for the binary group in $GF[p]$ to the binary group H with coefficients ranging over integers modulo p^2 and the determinant of transformation congruent to unity modulo p^2.

This thesis deals with the invariants of the linear homogeneous group in n variables with the coefficients ranging over integers modulo p^k, and the determinant of transformation congruent to unity modulo p.

The writer avails himself of this occasion to express his gratitude to Professor Dickson for the conduct of this research and for suggesting both the problem and the group-theoretic principles applied in the following pages.

1. PRELIMINARY

We shall consider linear homogeneous transformations (T) in indeterminates $x_1, x_2, \ldots, x_n$ with integral coefficients and having determinant of transformation $\equiv 1 \pmod{p}$.

The above transformations will be applied to forms in $x_1, x_2, \ldots, x_n$ with integral coefficients.

Two forms F_1 and F_2 will be considered identically congruent modulo p^k if and only if

$$F_1(x_1, x_2, \ldots, x_n) \equiv F_2(x_1, x_2, \ldots, x_n) + p^k F_3(x_1, x_2, \ldots, x_n)$$

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* Presented to the Society, April 13, 1923.

† These Transactions, vol. 12 (1911), pp. 75–98. Professor E. H. Moore constructed the invariant called L_n in the sequel.

algebraically, where F_3 is a form. The identity will be denoted by

$$F_1(x_1, x_2, \ldots, x_n) \equiv F_2(x_1, x_2, \ldots, x_n) \pmod{p^k}.$$

A formal invariant $F(x_1, x_2, \ldots, x_n)$ is a form which under all transformations

$$T: x_i = a_{i1}x_1' + a_{i2}x_2' + \cdots + a_{in}x_n' \qquad (i = 1, \ldots, n)$$

gives

$$F(x_1, x_2, \ldots, x_n) \equiv F(x_1', x_2', \ldots, x_n') \pmod{p^k},$$

after the new variables x_i' are replaced by their values in terms of the old variables obtained by solving the above n equations.

Two substitutions whose corresponding coefficients are congruent modulo p^k replace any form by two forms which are identically congruent, and which therefore play equivalent rôles in the theory of modular invariants. Accordingly, we shall say that two such substitutions belong to the same class.

In the sequel, for purposes of enumeration, we shall frequently represent a class by a "residual matrix" whose elements are least positive residues modulo p^k.

The classes of transformations can be put into one-to-one correspondence with the system of residual matrices, since to the compound of two classes corresponds the compound modulo p^k of the respective residual matrices.

The classes of substitutions T form a finite group; therefore the system of residual matrices forms a finite group of the same order under composition modulo p^k.

2. GROUP-THEORETIC INTRODUCTION

Let $G(n, p^k)$ be the total linear homogeneous group in n indeterminates with coefficients modulo p^k. Then the system of residual matrices may be represented by

$$\| a_{ij} + \alpha_{ij}^{(1)}p + \alpha_{ij}^{(2)}p^2 + \cdots + \alpha_{ij}^{(k-1)}p^{k-1} \| \qquad (i, j = 1, \ldots, n)$$

where the determinant of $\|a_{ij}\|$ is $\equiv 1 \pmod{p}$ and the elements $a_{ij}, \alpha_{ij}^{(1)}, \ldots, \alpha_{ij}^{(k-1)}$ range independently over the integers $0, 1, 2, \ldots, p-1$.

There are* $d = (p^n-1)(p^n-p)\cdots(p^n-p^{n-1})(p^n-p^{n-1})/(p-1)(p-1)$ sets of solutions of the determinantal congruence $|a_{ij}| \equiv 1 \pmod{p}$,

* L. E. Dickson, *Linear Groups*, p. 82.

and with every set we can associate $P = p^{n^2(k-1)}$ sets $\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots, \alpha_{ij}^{(k-1)}$. Therefore the order of the group $G(n, p^k)$ is dP.

In the sequel we shall consider two ways of decomposing $G(n, p^k)$ into a factor-group.

(1) Dividing every element in the matrices of the residual system by p, we obtain them in the form

$$\| b_{ij} + \beta_{ij} p \| \quad (i, j = 1, \ldots, n), \qquad |b_{ij}| \equiv 1 \pmod{p},$$

where b_{ij} ranges over the integers $0, 1, 2, \ldots, p-1$, and β_{ij} ranges over the integers $0, 1, 2, \ldots, p^{k-1}-1$.

The set of matrices $\| b_{ij} \|$ we shall denote by

$$H_1 \equiv (h_1, h_2, \ldots, h_m, \ldots, h_d).$$

The invariant subgroup of matrices $\| \beta_{ij} p \|$, with 1 added to the elements of the diagonal, we shall denote by

$$A_1 \equiv (a_1, a_2, \ldots, a_m, \ldots, a_P).$$

The group $G(n, p^k)$ may be decomposed into a factor-group of order d:

$$A_1 h_1, A_1 h_2, \ldots, A_1 h_m, \ldots, A_1 h_d.$$

The decomposition is exhaustive since no matrix of the subgroup A_1 is present in H_1, with the exception of the unit matrix, and the order of the subgroup A_1 multiplied by the number of matrices in H_1 gives the order of the group $G(n, p^k)$.

(2) Dividing every element in the matrices of the residual system by p^{k-1}, we obtain them in the form

$$\| c_{ij} + \gamma_{ij} p^{k-1} \| \quad (i, j = 1, \ldots, n), \qquad |c_{ij}| \equiv 1 \pmod{p},$$

where c_{ij} ranges over the integers $0, 1, 2, \ldots, p^{k-1}-1$, and γ_{ij} ranges over the integers $0, 1, 2, \ldots, p-1$.

The set of matrices $\| c_{ij} \|$ we shall denote by

$$H_{k-1} \equiv (h_1', h_2', \ldots, h_m', \ldots, h_\varkappa'), \qquad \varkappa = d p^{n^2(k-2)}.$$

The invariant abelian subgroup of matrices, $\|\gamma_{ij}p^{k-1}\|$, with 1 added to the elements of the diagonal, we shall denote by

$$A_{k-1} \equiv (a_1', a_2', \ldots, a_m', \ldots, a_{p^{n^2}}').$$

The group $G(n, p^k)$ may be decomposed into the factor-group

$$A_{k-1}h_1', A_{k-1}h_2', \ldots, A_{k-1}h_m', \ldots, A_{k-1}h_\varkappa', \qquad \varkappa = dp^{n^2(k-2)},$$

the exhaustiveness of the decomposition being proved as in (1), mutatis mutandis.

3. The invariants of the group $G(n, p^2)$

In this case the groups A_1 and A_{k-1}, and the sets H_1 and H_{k-1} coincide.

A comparison of the set H with the matrices of the group $G(n, p)$ shows that H has representatives of all the classes comprised by $G(n, p)$. Therefore the invariants of $G(n, p^2)$ may be put into the form

$$F(x_1, x_2, \ldots, x_n) + pF_1(x_1, x_2, \ldots, x_n), \tag{1}$$

where $F(x_1, x_2, \ldots, x_n)$ is an invariant of $G(n, p)$ and $F_1(x_1, x_2, \ldots, x_n)$ is some form.

As a preliminary step we shall inquire as to the conditions under which expressions of the type (1) are invariant under the sets of the factor-group

$$Ah_1, Ah_2, \ldots, Ah_m, \ldots, Ah_d.$$

Evidently (1) has to be invariant under the subgroup A.

Consider a general substitution of the subgroup A,

$$x_i' = x_i + p(\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n) \qquad (i = 1, \ldots, n).$$

On applying it to (1) we observe that $pF_1(x_1, x_2, \ldots, x_n)$ remains invariant under the transformation, because of the factor p. Therefore $F(x_1, x_2, \ldots, x_n)$ should be invariant under the subgroup A.

Applying the substitution to F by means of Taylor's expansion, we obtain

$$F(x_1', x_2', \ldots, x_n') \equiv F(x_1, x_2, \ldots, x_n) + p\sum_{i=1}^{n}(\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in}x_n)\frac{\partial F}{\partial x_i} \equiv 0 \pmod{p^2}.$$

($G(n, 2^2)$ is treated in Section 5.) This shows that F will be an invariant if the following congruence holds:

$$\sum_{i=1}^{n}(\alpha_{i1}x_1 + \alpha_{i2}x_2 + \cdots + \alpha_{in})\frac{\partial F}{\partial x_i} \equiv 0 \pmod{p}.$$

By taking $\alpha_{ii} = 1$ and the remaining $\alpha_{ij} = 0$, we get

$$x_i\frac{\partial F}{\partial x_i} \equiv 0 \pmod{p} \qquad (i = 1, \ldots, n),$$

whence we obtain the differential congruences, giving the necessary and sufficient conditions for invariance under A:

$$\frac{\partial F}{\partial x_1} \equiv \frac{\partial F}{\partial x_2} \equiv \cdots \equiv \frac{\partial F}{\partial x_n} \equiv 0 \pmod{p}. \tag{2}$$

Before proceeding with the application of conditions (2), we shall have need of more detailed information* as to the structure of the invariants of $G(n, p)$.

If we define

$$[e_1, e_2, \ldots, e_n] = \begin{vmatrix} x_1^{p^{e_1}} & x_2^{p^{e_1}} & \cdots & x_n^{p^{e_1}} \\ x_1^{p^{e_2}} & x_2^{p^{e_2}} & \cdots & x_n^{p^{e_2}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{p^{e_n}} & x_2^{p^{e_n}} & \cdots & x_n^{p^{e_n}} \end{vmatrix}$$

* L. E. Dickson, *A fundamental system of invariants*, these Transactions, vol. 12 (1911), p. 76.

then a fundamental system of invariants of $G(n,p)$ is given by the following set:

$$L_n = [n-1, n-2, \ldots, 1, 0],$$
$$Q_{ns} = \frac{[n, n-1, \ldots, s+1, s-1, \ldots, 1, 0]}{L_n} \qquad (s = 1, \ldots, n-1).$$

These invariants in terms of the invariants of $G(n-1,p)$ and L_n may be written

$$Q_{n1} = Q_{n-11}\left(\frac{L_n}{L_{n-1}}\right)^{p-1} + L_{n-1}^{p^2-p},$$
$$Q_{ns} = Q_{n-1s}\left(\frac{L_n}{L_{n-1}}\right)^{p-1} + Q_{n-1s-1}^{p} \qquad (s = 2, \ldots, n-2),$$
(3)
$$Q_{nn-1} = \left(\frac{L_n}{L_{n-1}}\right)^{p-1} + Q_{n-1n-2}^{p},$$
$$L_n = x_n L_{n-1}^{p} + L_{n-1}\sum_{s=1}^{n-2}(-1)^s x_n^{p^s} Q_{n-1s} + (-1)^{n-1} x_n^{p^{n-1}} L_{n-1}.$$

LEMMA. *No syzygy subsists among the invariants of $G(2,p)$, either algebraically or in the sense of a congruence.*

Assume the contrary, and arrange the polynomial in L_2 and Q_2 according to descending powers of Q_2; then we have

$$c_1 Q_2^{i_1} L_2^{j_1} + c_2 Q_2^{i_2} L^{j_2} + \cdots + c_m Q_2^{i_m} L_2^{j_m} = 0.$$

Here $c_1 Q_2^{i_1} L_2^{j_1}$ contains the highest power of x_1, for Q_2 contains a power of x_1 as a term, whereas L_2 contains only terms of the form $x_1^a x_2^b$. Therefore either the coefficients $c_1 = c_2 = \cdots = c_m = 0$, if we speak of an algebraic syzygy, or $c_1 \equiv c_2 \equiv \cdots \equiv c_m \equiv 0 \pmod{p^k}$, if we speak of it in the sense of a congruence.

THEOREM I.* *The invariants of $G(n,p)$ are not connected by a syzygy modulo p^k.*

Let the syzygy connecting the invariants be arranged according to powers of L_n:

$$\sum_m L_n^m \sum_{q_m} c_{q_m} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_m}} \equiv 0, \qquad c_{q_m} \not\equiv 0 \pmod{p^k}. \tag{4}$$

* Compare loc. cit., p. 83.

If m_0 is the lowest power of L_n in the above polynomial, we divide (4) by $L_n^{m_0}$ and obtain a sum of products not containing L_n as a factor:

$$\sum_{q_m} c_{q_m} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_m}} \text{ for } m = m_0. \tag{5}$$

All those terms which have L_n as a factor contain x_n, but (5) contains terms free of x_n whose sum therefore must be congruent to zero modulo p^k.

A reference to the preceding page shows that we obtain these terms by performing the substitution

$$S = \begin{pmatrix} Q_{n1} & Q_{n2} & \cdots & Q_{ns} & \cdots & Q_{nn-1} \\ L_{n-1}^{p^2-p} & Q_{n-11}^{p} & \cdots & Q_{n-1s-1}^{p} & \cdots & Q_{n-1n-2}^{p} \end{pmatrix}, \tag{6}$$

where distinct elements are replaced by distinct elements.

We obtain, therefore,

$$\sum_{q_m} c_{q_m} L_{n-1}^{(p^2-p)a_{1q_m}} \prod_{s=2}^{n-1} Q_{n-1s-1}^{pa_{sq_m}} \equiv 0 \pmod{p^k} \quad \text{for } m = m_0. \tag{7}$$

Since the products under the summation are formally distinct, (7) is a syzygy among the invariants of $G(n-1, p)$; this completes the induction, since by the lemma there is no syzygy connecting the invariants of $G(2, p)$.

THEOREM II. *Any form written as a polynomial in L_n and Q_{ns} ($s = 1, 2, \ldots, n-1$), satisfying the differential congruences (3), and not containing coefficients congruent to zero modulo p, involves L_n and Q_{ns} to powers whose exponents are multiples of p.*

Let

$$F(x_1, x_2, \ldots, x_n) \equiv \sum_m L_n^m \sum_{q_m} c_{q_m} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_m}},\ c_{q_m} = 0 \pmod{p}. \tag{8}$$

We shall first prove that $m \equiv 0 \pmod{p}$. By Euler's formula, we have

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} + \cdots + x_n \frac{\partial F}{\partial x_n} \equiv dF(x_1, x_2, \ldots, x_n) \pmod{p},$$

where d is the degree of the form F. Now since the left member is congruent to zero modulo p by (3), d must be divisible by p.

But

$$d \equiv m(p^{n-1}+\cdots+p+1)+\sum_{s=1}^{n-1} a_{sq_m}(p^n-p^s) \equiv 0 \pmod{p},$$

and since $\sum_{s=1}^{n-1} a_{sq_m}(p^n-p^s) \equiv 0 \pmod{p}$, m is divisible by p, whence $\frac{\partial F}{\partial L_n} \equiv 0 \pmod{p}$. Thus shall

$$\frac{\partial F}{\partial x_n} \equiv \frac{\partial F}{\partial L_n}\frac{\partial L_n}{\partial x_n}+\sum_{s=1}^{n-1}\frac{\partial F}{\partial Q_{ns}}\frac{\partial Q_{ns}}{\partial x_n} \equiv 0 \pmod{p}.$$

By the preceding, $\frac{\partial F}{\partial L_n} \equiv 0 \pmod{p}$, therefore it remains to consider

$$\sum_{s=1}^{n-1}\frac{\partial F}{\partial Q_{ns}}\frac{\partial Q_{ns}}{\partial x_n} \equiv 0 \pmod{p}. \tag{9}$$

Arranging it according to powers of L_n, we obtain

$$\frac{\partial F}{\partial x_n} \equiv \sum_m L_n^m \sum_{q_m} c_{q_m}\frac{\partial}{\partial x_n}\prod_{s=1}^{n-1} Q_{ns}^{a_{sq_m}} \equiv \Big(\sum_m L_n^m B_m\Big) C \equiv 0 \pmod{p},$$

where, for brevity,

$$B_m \equiv \sum_{q_m} c_{q_m}\left(\sum_{i=1}^{n-2} a_{iq_m} Q_{n-1\,i}\, Q_{ni}^{a_{iq_m}-1}\prod_{\substack{s=1\\ s\neq i}}^{n-1} Q_{ns}^{a_{sq_m}} + a_{n-1\,q_m} Q_{n\,n-1}^{a_{n-1\,q_m}-1}\prod_{s=1}^{n-2} Q_{ns}^{a_{sq_m}}\right)$$

$$C \equiv \frac{\partial\left(\frac{L_n}{L_{n-1}}\right)^{p-1}}{\partial x_n},$$

as follows by the use of (3).

Since by computation $C \not\equiv 0 \pmod{p}$, $\sum_m L_n^m B_m$ must be zero modulo p. Dividing it by the lowest power of L_n occurring there, i. e., with exponent $m = m_0$, we obtain a sum of products not containing L_n as a factor, i.e., B_{m_0}. B_{m_0} is the only part of $\dfrac{\sum_m L_n^m B_m}{L_n^{m_0}}$ free of the factor L_n. Now L_n itself contains x_n as a factor, therefore the only terms free of x_n are obtainable from B_{m_0}, and their sum should be congruent to zero modulo p.

We obtain this sum by performing substitution S (formula (6)) on B_{m_0}:

$$\sum_{q_m} c_{q_m} \left(\sum_{i=1}^{n-2} a_{iq_m} Q_{n-1\,i}^{p(a_{iq_m}-1)} Q_{n-1\,i-1}^{(p^2-p)a_{iq_m}} L_{n-1}^{(p^2-p)a_{iq_m}} \prod_{\substack{s=2 \\ s \neq i}}^{n-1} Q_{n-1\,s-1}^{pa_{sq_m}} \right.$$

$$\left. + a_{n-1\,q_m} Q_{n-1\,n-2}^{p(a_{n-1\,q_m}-1)} L_{n-1}^{(p^2-p)a_{1q_m}} \prod_{s=2}^{n-2} Q_{n-1\,s-1}^{pa_{sq_m}} \right) \equiv 0$$

$$(m = m_0) \pmod{p}.$$

Examining the products under the inner summation, we observe that they are distinct. In the ith product all the factors have as exponents multiples of p with the exception of $Q_{n-1\,i}$ which occurs raised to the $(pa_{sq_m}+1)$th power. The last product is the only product for a given q_m having all exponents of the factors multiples of p.

We thus see that the products arising from the differentiation of a certain $c_{q_m} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_m}} (m = m_0)$ in (8) are all distinct.

Two distinct products, as

$$c_{q_{1m}} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_{1m}}} \text{ and } c_{q_{2m}} \prod_{s=1}^{n-1} Q_{ns}^{a_{sq_{2m}}} \qquad (m = m_0),$$

could not give identical products on differentiation. An examination of the exponents would indicate that both were differentiated with respect to the same Q_{ns}, from the argument above. A substitution inverse to S (formula (6)) would lead to the same derivatives, and then we should arrive at the same primitives. The process is uniquely reversible.

All the products in the above are distinct and have as factors invariants of $G(n-1, p)$. But since no syzygy exists by the preceding theorem, the exponents appearing as coefficients $a_{sq_m}(m = m_0;\ s = 1, \ldots, n-1;$ for all $q_{m_0})$, are separately congruent to zero modulo p.

We can therefore divide by the next highest power of L_n, say $L_n^{m_1}$, and repeat the argument verbatim.

Therefore the a_{sq_m} are multiples of p for all s and q_m.

We thus see that $F(x_1, x_2, \ldots, x_n)$ must be a polynomial in L_n^p and Q_{ns}^p $(s = 1, \ldots, n-1)$ in order to be invariant under A. But this is also a sufficient condition for invariance under $G(n, p^2)$. Taking for example L_n^p and applying to it any substitutions of $G(n, p^2)$, we get

$$L_n^p(x_1', x_2', \ldots, x_n') \equiv \big(L_n(x_1, x_2, \ldots, x_n) + pf(x_1, x_2, \ldots, x_n)\big)^p$$

$$\equiv L_n^p(x_1, x_2, \ldots, x_n) \pmod{p^2},$$

where $f(x_1, x_2, \ldots, x_n)$ is some form.

Since $F(x_1, x_2, \ldots, x_n)$ is an invariant of $G(n, p^2)$, $pF_1(x_1, x_2, \ldots, x_n)$ must also be invariant under the group. However, operation upon a form pf with $G(n, p^2)$ is equivalent to operation upon f with $G(n, p)$, therefore $F_1(x_1, x_2, \ldots, x_n)$ must be a polynomial in L_n and $Q_{ns}(s = 1, \ldots, n-1)$.

We thus arrive at a fundamental system of invariants of the group $G(n, p^2)$, viz.,

$$L_n^p, \qquad Q_{ns}^p \quad (s = 1, \ldots, n-1), \qquad pL_n^a \prod_{s=1}^{n-1} Q_{ns}^{b_s},$$

where a and b_s may assume values from 0 to $p-1$, but may not all be zero.

4. Invariants of the group $G(n, p^k)$

A comparison of the matrices of the set H_{k-1} with those of the group $G(n, p^{k-1})$, shows that H_{k-1} has representatives of all the classes contained in $G(n, p^{k-1})$. Therefore the invariants of $G(n, p^k)$ may be put into form

$$P(x_1, x_2, \ldots, x_n) + p^{k-1} f(x_1, x_2, \ldots, x_n) \tag{1}$$

where $P(x_1, x_2, \ldots, x_n)$ is an invariant of $G(n, p^{k-1})$ and $f(x_1, x_2, \ldots, x_n)$ is some form.

We shall next investigate the form of expressions (1) which remain invariant under the sets of the factor-group

$$A_1 h_1, A_1 h_2, \ldots, A_1 h_m, \ldots, A_1 h_d,$$

d having the value indicated in Section 2.

Evidently they have to be invariant under the transformations of the invariant subgroup A_1.

A general substitution T_a of A_1 may be written

$$x_i' = x_i + p(\alpha_{i1} x_1 + \alpha_{i2} x_2 + \cdots + \alpha_{in} x_n) \qquad (i = 1, \ldots, n).$$

We have to apply the substitution only to $P(x_1, x_2, \ldots, x_n)$, since $p^{k-1} f(x_1, x_2, \ldots, x_n)$ is invariant under the subgroup A_1.

Since $P(x_1, x_2, \ldots, x_n)$ is an invariant of $G(n, p^{k-1})$, we shall assume, for purposes of induction, that it may be written in the form

$$P(x_1, x_2, \ldots, x_n) \equiv F_0 + pF_1 + p^2 F_2 + \cdots + p^{k-2} F_{k-2} \tag{2}$$

where

$$F_j \text{ is a polynomial in } Q_{ns}^{p^{k-j-2}},\ L^{p^{k-j-2}} \qquad \begin{matrix}(s = 1, \ldots, n-1)\\ (j = 0, \ldots, k-2)\end{matrix} \tag{3}$$

not involving coefficients congruent to zero modulo p.

From (3) it is clear that

$$\frac{\partial F_j}{\partial x_i} \equiv \frac{\partial F_j}{\partial L_n}\frac{\partial L_n}{\partial x_i} + \sum_{s=1}^{n-1} \frac{\partial F_j}{\partial Q_{ns}}\frac{\partial Q_{ns}}{\partial x_i} \equiv 0 \pmod{p^{k-j-2}} \tag{4}$$
$$(j = 0, \ldots, k-2).$$

Also it is obvious that for the case of polynomials with integral exponents

$$\frac{\partial F_j}{\partial x_i} \equiv 0 \pmod{p^{k-j-2}} \text{ implies } \frac{\partial^r F_j}{\partial x_i^r} \equiv 0 \pmod{p^{k-j-2}}.$$

The application of T_a to $P(x_1, x_2, \ldots, x_n)$ gives by Taylor's expansion

$$(5)\quad P(x_1', x_2', \ldots, x_n') \equiv P(x_1, x_2, \ldots, x_n) + \frac{1}{r!}\sum_{r=1}^{m}\sum_{j=0}^{k-2} p^{j+r}\left[\sum_{i=1}^{n}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)\frac{\partial F_j}{\partial x_i}\right]^r,$$

r having the usual symbolical significance as a power, and m being the highest degree in any one variable of the polynomial of the highest order.

Since $F_j\,(j = 0, \ldots, k-2)$ are polynomials, the coefficients of the expansions are integers.

Now by (4), since $\dfrac{\partial F_j}{\partial x_i} \equiv 0 \pmod{p^{k-j-2}}$, we see that

$$(6)\quad \frac{1}{r!}\sum_{r=2}^{m}\sum_{j=0}^{k-2} p^{j+r}\left(\sum_{i=1}^{n}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)\frac{\partial F_j}{\partial x_i}\right)^r \equiv 0 \pmod{p^k}$$

for all odd primes. The case $p = 2$ will be considered separately.

From (5) and (6) it follows that $P(x_1, x_2, \ldots, x_n)$ will be an invariant of subgroup A_1 if

$$\sum_{j=0}^{k-2}\sum_{i=1}^{n} p^{j+1}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)\frac{\partial F_j}{\partial x_i} \equiv 0 \pmod{p^k}.$$

By taking $a_{ii} = 1$ and the remaining $a_{ij} = 0$, we obtain the equivalent set of n differential congruences

$$(7)\quad \sum_{j=0}^{k-2} p^{j+1}\frac{\partial F_j}{\partial x_i} \equiv 0 \pmod{p^k} \qquad (i = 1, \ldots, n).$$

We shall obtain now additional information about $P(x_1, x_2, \ldots, x_n)$ from the consideration of its homogeneity in $x_1, x_2, \ldots, x_n$.

If the power of L_n and Q_{ns} in any one term of F_j be $a_j\,p^{k-j-2}$ and $b_{js}\,p^{k-j-2}$, by recalling that the degree of L_n is $(p^{n-1} + \cdots + p + 1)$ and that of Q_{ns} is $(p^n - p^s)$, we obtain the following set of equations:

$$a_0 p^{k-2}(p^{n-1}+\cdots+p+1)+\sum_{s=1}^{n-1} b_{0s} p^{k-2}(p^n-p^s)$$

$$= a_1 p^{k-3}(p^{n-1}+\cdots+p+1)+\sum_{s=1}^{n-1} b_{1s} p^{k-3}(p^n-p^s)$$

$$= \cdots\cdots$$

$$= a_{k-2}(p^{n-1}+\cdots+p+1)+\sum_{s=1}^{n-1} b_{k-2s}(p^n-p^s),$$

from which it follows that

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{k-2} \equiv 0 \pmod{p}.$$

LEMMA 1. *In this way we see that L_n occurs in F_j $(j = 1, \ldots, k-2)$ raised to powers which are multiples of p^{k-j-1}, and therefore*

$$\sum_{j=1}^{k-2} p^{j+1} \frac{\partial F_j}{\partial L_n} \equiv 0 \pmod{p^k}.$$

This considerably simplifies the differential congruences (7), which we shall write as follows:

$$(8)\quad p \frac{\partial F_0}{\partial L_n}\frac{\partial L_n}{\partial x_i}+\sum_{j=0}^{k-2}\sum_{s=1}^{n-1} p^{j+1}\frac{\partial F_j}{\partial Q_{ns}}\frac{\partial Q_{ns}}{\partial x_i} \equiv 0 \pmod{p^k} \quad (i = 1, \ldots, n).$$

Multiplying the ith congruence by x_i, adding, and applying Euler's formula for homogeneous functions, we obtain from (8)

$$p \frac{\partial F_0}{\partial L_n} L_n \equiv 0 \pmod{p^k},$$

since the degree of Q_{ns} is divisible by p.

LEMMA 2. *By the proof of Theorem* I, *this shows that L_n occurs in F_0 raised to powers which are multiples of p^{k-1}.*

From the preceding result, we obtain (8) in the following simplified form:

$$(9) \qquad \sum_{j=0}^{k-2} \sum_{s=1}^{n-1} p^{j+1} \frac{\partial F_j}{\partial Q_{ns}} \frac{\partial Q_{ns}}{\partial x_i} \equiv 0 \pmod{p^k} \qquad (i = 1, \ldots, n).$$

We shall confine our attention to the nth congruence and prove that Q_{ns} $(s = 1, \ldots, n-1)$ must appear in F_j with an exponent which is congruent to zero modulo p^{k-j-1}.

The proof of Theorem II from the point where the differential congruence is simplified to the form (9) allows us to state the following theorem without proof:

THEOREM III. *The polynomial* $\sum_{j=0}^{k-2} F_j$, *defined by* (3), *Section* 4, *satisfying the congruence*

$$\sum_{j=0}^{k-2} \sum_{s=1}^{n-1} p^{j+1} \frac{\partial F_j}{\partial Q_{ns}} \frac{\partial Q_{ns}}{\partial x_n} \equiv 0 \pmod{p_k}$$

involves Q_{ns} $(s = 1, \ldots, n-1)$ *raised to powers which are multiples of* p.

But this will give $\sum_{s=1}^{n-1} p^{k-1} \frac{\partial F_{k-2}}{\partial Q_{ns}} \equiv 0 \pmod{p^k}$, and enable us to restate Theorem III in the following way:

The polynomial $\sum_{j=0}^{k-3} F_j$, *defined by* (3), *Section* 4, *satisfying the congruence*

$$\sum_{j=0}^{k-3} \sum_{s=1}^{n-1} p^{j+1} \frac{\partial F_j}{\partial Q_{ns}} \frac{\partial Q_{ns}}{\partial x_n} \equiv 0 \pmod{p_k}$$

involves Q_{ns} $(s = 1, \ldots, n-1)$ *raised to powers which are multiples of* p^2.

This theorem in its turn will yield the result

$$\sum_{s=1}^{n-1} p^{k-2} \frac{\partial F_{k-3}}{\partial Q_{ns}} \equiv 0 \pmod{p^k}.$$

Proceeding in this way we arrive at the result that F_j involves Q_{ns} $(s = 1, \ldots, n-1)$ raised to powers which are congruent to zero modulo p^{k-j-1}.

Collecting results, we obtain from Lemmas 1, 2, Theorem III and sequel, that F_j is a polynomial in $L_n^{p^{k-j-1}}$ and $Q_{ns}^{p^{k-j-1}}$.

Conversely, exactly as at the conclusion of Section 3, we can prove that $p^j L_n^{p^{k-j-1}}$ and $p^j Q_{ns}^{p^{k-j-1}}$ $(s = 1, \ldots, n-1)$, for $j = 0, \ldots, k-1$, are invariants of the group $G(n, p^k)$.

Thus a fundamental system of invariants of group $G(n, p^k)$ is given by the following set:

$$L_n^{p^{k-1}},\ Q_{ns}^{p^{k-1}}\ (s = 1, \ldots, n-1),\ p^j L_n^{ap^{k-j-1}} \prod_{s=1}^{n-1} Q_{ns}^{b_s p^{k-j-1}} \quad (j = 1, \ldots, k-1),$$

where a and b_s range over $0, 1, \ldots, p-1$, but may not all be zero.

5. Invariants of the group $G(n, 2^k)$

In this case, which was not considered in Section 4, but was alluded to in connection with congruence (6), the condition for invariance under the subgroup A_1 takes the form

$$\sum_{j=0}^{k-2} 2^{j+1} \sum_{i=1}^{n} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) \frac{\partial F_j}{\partial x_i} + \sum_{j=0}^{k-2} 2^{j+1} \left[\sum_{i=1}^{n} (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n) \frac{\partial F_j}{\partial x_i} \right]^2 \equiv 0 \pmod{2^k}.$$

By taking $a_{ii} = 1$ and the remaining $a_{ij} = 0$, the above simplifies into n congruences,

$$(1) \qquad \sum_{j=0}^{k-2} 2^{j+1} \frac{\partial F_j}{\partial x_i} + \sum_{j=0}^{k-2} 2^{j+1} x_i \frac{\partial^2 F_j}{\partial x_i^2} \equiv 0 \pmod{2^k} \qquad (i = 1, \ldots, n),$$

by dividing each congruence by x_i.

If we employ the following sets of values for a_{ij}, $a_{kk+1} = 1$ and the remaining $a_{ij} = 0$ (the subscripts being taken modulo n), we obtain, by dividing through respectively by $x_2, x_3, \ldots, x_n, x_1$, the n congruences

$$(2) \qquad \sum_{j=0}^{k-2} 2^{j+1} \frac{\partial F_j}{\partial x_i} + \sum_{j=0}^{k-2} 2^{j+1} x_{i+1} \frac{\partial^2 F_j}{\partial x_i^2} \equiv 0 \pmod{2^k} \qquad (i = 1, \ldots, n),$$

where the subscript of x_{i+1} in the second term is taken modulo n.

By subtracting the ith congruence of (2) from the ith congruence of (1), we obtain the n relations

$$(x_i - x_{i+1}) \sum_{j=0}^{k-2} 2^{j+1} \frac{\partial^2 F_j}{\partial x_i^2} \equiv 0 \pmod{2^k} \qquad (i = 1, \ldots, n).$$

Since $x_i - x_{i+1} \not\equiv 0 \pmod{2^k}$, we must have

$$\sum_{j=0}^{k-2} 2^{j+1} \frac{\partial^2 F_j}{\partial x_i^2} \equiv 0 \pmod{2^k} \qquad (i = 1, \ldots, n). \tag{3}$$

The congruential identities (3) when substituted in (1) give

$$\sum_{j=0}^{k-2} 2^{j+1} \frac{\partial F_j}{\partial x_i} \equiv 0 \pmod{2^k} \qquad (i = 1, \ldots, n).$$

i. e., exactly the results (7) Section 4 for modulus 2^k.

Therefore all the conclusions arrived at in Section 4 for powers of odd prime moduli hold for 2^k.

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ON CERTAIN POLAR CURVES WITH THEIR APPLICATION TO THE LOCATION OF THE ROOTS OF THE DERIVATIVES OF A RATIONAL FUNCTION*

BY

B. Z. LINFIELD

INTRODUCTION

The question asked by Professor Maxime Bôcher† in a discussion concerning certain theorems on the roots of the derivative of a polynomial, "Could not the first of these propositions be brought into connection with the focal properties of the higher plane curves?" had been answered earlier by M. van den Berg‡ with the following theorem:§ *Les points racines de la dérivée d'une équation à n racines différentes sont les* $n-1$ *foyers d'une courbe de* $(n-1)$*e classe qui touche dans leurs milieux les* $\frac{1}{2}n(n-1)$ *côtes du n-agone complet déterminé par l'équation originale. L'équation originale ayant des racines multiples, la droite joignant un point-racine* m^{ple} *à un point-racine* p^{ple} *est divisée à raison de m à p par le point de contact.* An independent proof of this theorem was given also by P. J. Heawood for the polynomial with simple roots, and the extension to the case of multiple roots was indicated‖. M. Fujiwara¶ generalized van den Berg's theorem as follows: *Die Wurzelpunkte von* $f^k(x) = 0$ *sind die Brennpunkte der kten polaren Kurve* $(n-k$*ter Klasse*$)$ *der Kurve nter Klasse, welche aus den n Wurzelpunkte von* $f(x) = 0$ *besteht, in bezug auf die unendlich-ferne Gerade.* Apparently the last writer dealt with a polynomial of simple roots, and he therefore failed to note the relationship of his kth polar

* Presented to the Society October 28, 1922.

† Annals of Mathematics, vol. 7 (1892), p. 70.

‡ Nieuw Archief voor Wiskunde, vol. 15 (1888), p. 190.

§ The theorem is stated as quoted by M. Jan de Vries, Rendiconti del Circolo Matematico di Palermo, vol. 5 (1891), p. 290.

‖ Quarterly Journal of Mathematics, vol. 38 (1907), p. 80.

The writer in ignorance of the contents of these papers and in the belief that Bôcher's question was unanswered, proved this theorem for the rational function of multiple roots and poles, Bulletin of the American Mathematical Society, vol. 27 (1920), p. 17. He is indebted to Professor A. J. Kempner and to Dr. J. L. Walsh for the references to van den Berg and Heawood. The investigation was undertaken at the suggestion of Professor W. H. Echols of the University of Virginia to whom the writer feels greatly indebted for encouragement and inspiration.

¶ Tôhôku Mathematical Journal, vol. 9 (1916), p. 108.

curve to the original polynomial of the roots of $f(x) = 0$, as van den Berg has done in the case of the first derivative of a polynomial, and as the writer will do for the case of the rth derivative of a rational function of multiple zeros and poles. Some theorems concerning the properties of these curves will then be taken up, from which the foci will be localized to limited areas, and applications will be made to particular functions where it will be shown that the curves can be traced with ease.

PART I

1. NOTATION AND PRELIMINARY LEMMAS

The operation of polarization with respect to a line of a curve given in homogeneous, rectangular, line-coördinates is well known. The rth polar of $F(u, v, w) = 0$ with respect to the line (u', v', w') is given, in the usual symbolic notation, by

$$\left(u' \frac{\partial}{\partial u} + v' \frac{\partial}{\partial v} + w' \frac{\partial}{\partial w}\right)^r F = 0.$$

We propose to extend the name "curve" to a homogeneous rational fractional function of u, v, w (we shall then write curve in italics), and to apply to it the operation of polarization with respect to a line. If $F(u, v, w) \equiv F_1(u, v, w)/F_2(u, v, w)$, where $F_1(u, v, w)$ and $F_2(u, v, w)$ are homogeneous polynomials in u, v, w, having no common factor that is not a constant, then, as before, the rth polar of the *curve* $F(u, v, w) = 0$ with respect to the line (u', v', w') is given by

$$\left(u' \frac{\partial}{\partial u} + v' \frac{\partial}{\partial v} + w' \frac{\partial}{\partial w}\right)^r F = 0.$$

In particular, if the line (u', v', w') be the line at infinity $(u' = v' = 0)$, then the rth polar of $F(u, v, w) = 0$ is $\dfrac{\partial^r}{\partial w^r}\left[F_1(u, v, w)/F_2(u, v, w)\right] = 0$, which we designate, for the sake of brevity, by $F_w^r = 0$. The numerator of F_w^r* we denote by (F_w^r). With these explanations clearly in mind, we can prove the following lemmas concerning the curve $(F_w^r) = 0$.

LEMMA 1. *If $F_1(u, v, w) = 0$ touches the X-axis in k_1 points (not necessarily distinct), given by the roots of $\theta_1(x) = 0$, and if $F_2(u, v, w) = 0$ touches the X-axis in k_2 distinct and different points given by the roots of $\theta_2(x) = 0$, the*

* The numerator and the denominator of F_w^r have no common factor other than a constant.

points of contact of $(F_w^r) = 0$ *with the X-axis are given by the roots of the rth derivative of* $\theta_1(x)/\theta_2(x) = 0$.*

Case 1. Suppose $F_2(u, v, w)$ has no multiple factors.

Let the roots of $\theta_1(x) = 0$ and of $\theta_2(x) = 0$ be $x_1, \ldots, x_{k_1}$ and $x_1', \ldots, x_{k_2}'$ respectively, where $x_1' \neq x_2' \neq \cdots \neq x_{k_2}'$ and $\theta_1(x_i') \neq 0$ $(i = 1, 2, \ldots, k_2)$. We can write therefore

$$\theta_1(x) \equiv \prod_{i=1}^{k_1}(x - x_i) = 0, \quad \text{and} \quad \theta_2(x) \equiv \prod_{i=1}^{k_2}(x - x_i') = 0.$$

If the degree of $F_1(u, v, w)$ is n_1, then, in order that the curve $F_1(u, v, w) = 0$ should touch the X-axis in the points given by

$$\prod_{i=1}^{k_1}(x - x_i) = 0,$$

we must have

$$(1) \qquad F_1(u, v, w) = v^{n_1 - k_1}\prod_{i=1}^{k_1}(ux_i + w) + \Phi_1(u, v, w) = 0,$$

where the highest power of v in $\Phi_1(u, v, w)$ is less than $n_1 - k_1$; for the coördinates of the points of contact of $F_1(u, v, w) = 0$ with the X-axis are the values of $-w/u$ in $F_2(u, v, w) = 0$ when v approaches infinity, and these values are the roots of the coefficient of the highest power of v equated to zero when the latter is considered as an equation in $-w/u$. Similarly, if n_2 is the degree of $F_2(u, v, w) = 0$, we have

$$(2) \qquad F_2(u, v, w) = v^{n_2 - k_2}\prod_{i=1}^{k_2}(ux_i' + w) + \Phi_2(u, v, w) = 0,$$

where the highest coefficient of v in $\Phi_2(u, v, w)$ is less than $n_2 - k_2$. To find the points of contact of $(F_w^r) = 0$ with the X-axis, we equate the coefficient of the highest power of v in the numerator of $(\partial^r/\partial w^r)F(u, v, w)$ to zero. Since

$$F(u, v, w) \equiv \frac{F_1(u, v, w)}{F_2(u, v, w)},$$

* When the derivative has infinite roots arising from the vanishing of the highest coefficient or coefficients of the numerator, homogeneous coördinates should be used.

we have

$$\frac{\partial^r F}{\partial w^r} = F_1 \frac{\partial^r (1/F_2)}{\partial w^r} + r \frac{\partial F_1}{\partial w} \frac{\partial^{r-1}(1/F_2)}{\partial w^{r-1}} + \cdots + \frac{\partial^r (1/F_2)}{\partial w^r}.$$

But we know*

$$\frac{\partial^r (1/F_2)}{\partial w^r} = (-1)^r \frac{r!}{F_2^{r+1}} \Delta^r \{F_2\},$$

where

$$\Delta^r \{F_2\} \equiv \begin{vmatrix} \frac{\partial F_2}{\partial w} & F_2 & 0 & 0 & \cdots \\ \frac{\partial^2 F_2}{\partial w^2} \cdot \frac{1}{2!} & \frac{\partial F_2}{\partial w} & F_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^r F_2}{\partial w^r} \cdot \frac{1}{r!} & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial w} \end{vmatrix}.$$

Hence we have the formula

$$(3) \quad (F_w^r) = [F_2]^r \frac{\partial^r F_1}{\partial w} - \frac{r!}{(r-1)!} [F_2]^{r-1} \frac{\partial^{r-1} F_1}{\partial w^{r-1}} \Delta^1 \{F_2\} + \cdots$$

$$\cdots + (-1)^r r! F_1 \Delta^r \{F_2\},$$

as there is no factor common to the expression on the right of equation (3) and to $[F_2]^{r+1}$. If we substitute for F_1 and F_2 in the last equation their values given by equations (1) and (2), we see that the coefficient of the highest power of v in this equation, i. e., the coefficient of $v^{n_1-k_1+r(n_2-k_2)}$, is

$$(4) \quad \prod_{i=1}^{k_2} (ux_i' + w)^r \frac{\partial^r}{\partial w^r} \prod_{i=1}^{k_1} (ux_i + w)$$

$$- \frac{r!}{(r-1)!} \prod_{i=1}^{k_2} (ux_i' + w)^{r-1} \frac{\partial^{r-1}}{\partial w^{r-1}} \prod_{i=1}^{k_1} (ux_i + w) - \Delta^1 \left\{ \prod_{i=1}^{k_2} (ux_i' + w) \right\} + \cdots,$$

* Echols, *Calculus*, p. 455.

which is clearly the numerator of

$$\frac{\partial^r}{\partial w^r}\left[\prod_{i=1}^{k_1}(ux_i+w)\Big/\prod_{i=1}^{k_2}(ux_i'+w)\right]. \tag{5}$$

The points of contact of $(F_w^r)=0$ with the X-axis are the roots of the numerator of (5) considered as a function of $-w/u$ and equated to zero. Hence designating $-w/u$ by x, we have the points of contact of $(F_w^r)=0$ with the X-axis given by the roots of the rth derivative of $\theta_1(x)/\theta_2(x)=0$.

Case 2. $F_2(u,v,w)$ has multiple factors.

Since $F_2(u,v,w)$ has multiple factors, it is of the form

$$F_2(u,v,w)=\prod_{j=1}^{s}[v^{\sigma_j}+\Phi_{\sigma_j}(u,v,w)]^{t_j}\left[v^{n_2-k_2-\sigma}\prod_{i=1}^{k_2}(ux_i'+w)+\Phi_{n_2-\sigma}(u,v,w)\right]^{*} \tag{6}$$

where $t_j\geqq 2$, $\sigma=\sum_{j=1}^{s}t_j\sigma_j$, $n_2-k_2\geqq\sigma$, and $v^{n_2-k_2-\sigma}\prod_{i=1}^{k_2}(ux_i'+w)+\Phi_{n_2-\sigma}$ has no multiple factors, for the coefficient of the highest power of v in $F_2(u,v,w)$ must be $\prod_{i=1}^{k_2}(ux_i'+w)$, which has no multiple factors. The expression on the right of equation (3), therefore, has a factor in common with (F_w^r), namely

$$\prod_{j=1}^{s}[v^{\sigma_j}+\Phi_{\sigma_j}(u,v,w)]^{r(t_j-1)}{}^{\dagger}, \tag{7}$$

and (F_w^r) is equal to the expression on the right of equation (3) divided by (7). But this does not affect the identity of the coefficients of the highest power of v in (F_w^r) and the numerator of (5). Hence, as before, the points of contact of $(F_w^r)=0$ with the X-axis are given by the roots of the rth derivative of $\theta_1(x)/\theta_2(x)=0$, and the lemma is established.

In the following lemmas we designate $\theta_1(x)/\theta_2(x)$ by $\theta(x)$, and the rth derivative of $\theta(x)$ by $\theta^r(x)$.

---

* The subscript of Φ designates its degree.

† The degree of $(F_w^r)=0$ is therefore equal to $n_1+r\left(n_2-\sigma+2\sum_{j=1}^{s}\sigma_j-1\right)$.

LEMMA 2*. *If not all of the points of contact of* $F_2(u, v, w) = 0$ *with the X-axis are distinct and different from the points of contact of* $F_1(u, v, w) = 0$ *with the same axis, i. e.,*

$$\theta_2(x) = \prod_{i=1}^{p}(x-x_i')^{\mu_i}\prod_{i=p+1}^{q}(x-x_j')^{\mu_j}\prod_{k=q+1}^{s}(x-x_k') = 0 \quad (\mu_i \geqq 1,\ \mu_j \geqq 2),$$

$$\theta_1(x_i') = 0 \ (i = 1, 2, \ldots, p), \text{ and } \theta_1(x_i') \neq 0 \ (i = p+1, \ldots, s),$$

and if $F_2(u, v, w) = 0$ *has multiple factors such that*

$$F_2(u, v, w) = \prod_{i=1}^{\tau}[v^{\sigma_i}\Phi_{\rho_i}(u, v) + \Phi_{\sigma_i+\rho_i}(u, v, w)]^{t_i}\Phi(u, v, w) = 0,$$

where the highest power of v *in* $\Phi_{\sigma_i+\rho_i}(u, v, w)$ *is less than* σ_i, $t_i \geqq 2$, *and where* $\Phi(u, v, w)$ *has no multiple factors, the points of contact of* $(F_w^r) = 0$ *with the X-axis are given by the roots of*

$$\frac{\left[\prod_{i=1}^{q}(x-x_i')^{\mu_i}\right]^{r+1}}{\prod_{i=1}^{\tau}[\Phi_{\rho_i}(-1, x)]^{r(t_i-1)}}\theta^r(X) = 0,\dagger$$

provided $\theta^r(x) \not\equiv 0$.

The proof of this lemma is very similar to the proof of the previous one, and we therefore do not repeat it. We only recall that (F_w^r), under the conditions of this lemma, is equal to the expression on the right of equation (3) divided by $\prod_{i=1}^{\tau}[v^{\sigma_i}\Phi_{\rho_i}(u, w) + \Phi_{\sigma_i+\rho_i}(u, v, w)]^{r(t_i-1)}$.

COROLLARY 1. *If* $p = 0$, $\tau = q$, $\sigma_i = 0$, $t_i = \mu_i$, *and* $\Phi_{\rho_i}(u, w) \equiv ux_i' + w$, *then the points of contact of* $(F_w^r) = 0$ *with the X-axis are again the roots of* $\theta^r(x) = 0$.

LEMMA 3. *If the x-intercepts of the tangents of* $F_1(u, v, w) = 0$ *of slope* m *are given by the roots of* $\theta_1(x) = 0$, *and if the x-intercepts of the tangents of* $F_2(u, v, w) = 0$ *of the same slope are given by the roots of* $\theta_2(x) = 0$ *which*

* While this lemma is more general than we need, the proof of the less general one is not much simpler.

† An infinite value of the denominator is not here considered a root.

are distinct and different from the roots of $\theta_1(x) = 0$, *the x-intercepts of the tangents of* $(F_w^r) = 0$ *of slope m are given by the roots of* $\theta^r(x) = 0$.

LEMMA 4. *If not all the tangents of* $F_2(u, v, w) = 0$ *of slope m are distinct and different from the tangents of* $F_1(u, v, w) = 0$ *of the same slope, i.e.,*

$$\theta_2(x) = \prod_{i=1}^{p}(x-x_i)^{\mu_i}\prod_{j=p+1}^{q}(x-x_j)^{\mu_j}\prod_{k=q+1}^{s}(x-x_k) = 0 \quad (\mu_i \geqq 1,\ \mu_j \geqq 2),$$

$$\theta_1(x_i) = 0 \ (i = 1, 2, \ldots, p), \quad \text{and} \quad \theta_1(x_i) \neq 0 \ (i = p+1, \ldots, s),$$

and if $F_2(u, v, w) = 0$ *has multiple factors such that*

$$F_2(u, v, w) \equiv \prod_{i=1}^{\tau}[\Phi_{\rho_i}(u, w) + v\,\Phi_{\rho_i - 1}(u, v, w)]^{t_i}\,\Phi(u, v, w) = 0,$$

where $\Phi(u, v, w)$ *has no multiple factor, and where* $t_i \geqq 2$, *the x-intercepts of the tangents of* $(F_w^r) = 0$ *with the slope m are given by the roots of*

$$\frac{\left[\prod_{i=1}^{q}(x-x_i)^{\mu_i}\right]^{r+1}}{\prod_{i=1}^{\tau}[\Phi_{\rho_i}(-1, x)]^{r(t_i-1)}}\,\theta^r(x) = 0,$$

provided $\theta^r(x) \not\equiv 0$.

COROLLARY 1. *If* $p = 0$, $\tau = q$, $t_i = \mu_i$ *and* $\Phi_{\rho_i}(u, w) \equiv ux_i + w$ $(i = 1, \ldots, q)$, *the x-intercepts of the parallel tangents of* $(F_w^r) = 0$ *are again the roots of* $\theta^r(x) = 0$.

LEMMA 5. *If the foci of* $F_1(u, v, w) = 0$ *and of* $F_2(u, v, w) = 0$ *are given in the complex plane by the roots of the polynomials* $f_1(z) = 0$ *and* $f_2(z) = 0$* *respectively, such that the roots of* $f_2(z) = 0$ *are distinct and different from the roots of* $f_1(z) = 0$, *the foci of* $(F_w^r) = 0$ *are the roots of the rth derivative of* $f_1(z)/f_2(z) = 0$.

---

* The foci of $F_1(u, v, w) = 0$ and of $F_2(u, v, w) = 0$ are here assumed to be finite. The extension of this lemma to include the case of infinite foci is made obvious by the introduction of homogeneous complex variables. We have no occasion here to use the extension. The proofs of these lemmas are similar to the proof of Lemma 1 and are omitted. See Emch, Bulletin of the American Mathematical Society, vol. 25 (1917), p. 157.

LEMMA 6. *If not all the foci of* $F_2(u, v, w) = 0$ *are distinct and different from the foci of* $F_1(u, v, w) = 0$, *i.e.,*

$$f_2(z) = \prod_{i=1}^{p} (z-z_i)^{\mu_i} \prod_{j=1}^{q} (z-z_j)^{\mu_j} \prod_{k=q+1}^{s} (z-z_k) = 0 \quad (\mu_i \geqq 1,\ \mu_j \geqq 2),$$

$$f_1(z_i) = 0 \ (i = 1, \ldots, p), \qquad f_1(z_i) \neq 0 \ (i = p+1, \ldots, s);$$

and if $F_2(u, v, w) = 0$ *has multiple factors such that*

$$F_2(u, v, w) = \prod_{i=1}^{\tau} [\Phi_{\rho_i}(u, v, w)]^{t} \ \Phi(u, v, w) = 0,$$

where $t_i \geqq 2$ *and where* $\Phi(u, v, w)$ *has no multiple factors, the foci of* $(F_w^r) = 0$ *are the roots of*

$$\frac{\left[\prod_{i=1}^{q} (z-z_i)^{\mu_i}\right]^{r+1}}{\prod_{i=1}^{\tau} [\Phi_{\rho_i}(-1, -i, z)]^{r(t_i-1)}} f^r(z) = 0,$$

provided $f^r(z) \not\equiv 0$.

COROLLARY 1. *If* $p = 0$, $\tau = q$, $t_i = \mu_i$, *and* $\Phi_{\rho_i}(-1, -i, z) \equiv z - z_i$ $(i = 1, 2, \ldots, q)$, *the foci of* $(F_w^r) = 0$ *are again the roots of* $f^r(z) = 0$.

2. Generalization of van den Berg's theorem

Let

$$f(z) \equiv \prod_{i=1}^{n} (z-z_i)^{\mu_i} \qquad (z_i \neq z_j,\ \mu_i \leqq \mu_{i+1},\ i, j = 1, \ldots, n-1)$$

be a rational function of z. Let k be the number of distinct poles of $f(z)$, i.e. $\mu_i < 0$, $i = 1, \ldots, k$; $\mu_i > 0$, $i = k+1, \ldots, n$. We consider the *curve*

$$\varphi(u, v, w) \equiv \prod_{i=1}^{n} \alpha_i^{\mu_i} = 0 \qquad (z_i : \alpha_i \equiv ux_i + vy_i + w).$$

Clearly it is of the type $F(u, v, w) = 0$ considered above, where

$$F_1(u, v, w) \equiv \prod_{i=k+1}^{n} \alpha_i^{\mu_i} = 0, \text{ and } F_2(u, v, w) \equiv \prod_{i=1}^{k} \alpha_i^{|\mu_i|} = 0.$$

The foci of $\prod_{i=k+1}^{n} \alpha_i^{\mu_i} = 0$ are the roots of $\prod_{i=k+1}^{n} (z - z_i)^{\mu_i} = 0$, those of $\prod_{i=1}^{k} \alpha_i^{|\mu_i|} = 0$ are the roots of $\prod_{i=1}^{k} (z - z_i)^{|\mu_i|} = 0$, and those of $(\varphi_w^r) = 0$ are given by Corollary 1, Lemma 4. Any line through $\alpha_i = 0$ $(i \leqq k)$ is tangent to $\prod_{i=1}^{k} \alpha_i^{|\mu_i|} = 0$ at the point z_i. Similarly any line through $\alpha_j = 0$ $(k < j \leqq n)$ is tangent to $\prod_{i=k+1}^{n} \alpha_i^{\mu_i} = 0$ at the point z_j. If m_1 of the $n - k$ zeros are collinear with m_2 of the k poles, and if no other zero of $f(z)$ is collinear with them, then the points of contact of $(\varphi_w^r) = 0$ with that line are given by Corollary 1, Lemma 2. Hence:

THEOREM 1. *The zeros of the rth derivative of the rational function*

$$f(z) \equiv \prod_{i=1}^{n} (z - z_i)^{\mu_i}$$

are the foci of $(\varphi_w^r) = 0$, *where*

$$\varphi(u, v, w) \equiv \prod_{i=1}^{n} \alpha_i^{\mu_i} \quad (z_i : \alpha_i \equiv ux_i + vy_i + w).$$

The curve $(\varphi_w^r) = 0$ *touches the line through* m *zeros and poles of* $f(z)$ $(m < n)$* *in points which are independent of the remaining zeros and poles of* $f(z)$, *provided no other zero or pole of* $f(z)$ *lies on that line. If this line be taken for the X-axis, the points of contact are given by the roots of the rth derivative of*

$$\prod_{i=1}^{m} (x - x_i)^{\mu_i} = 0,$$

* When $m = n$ the curve $(\varphi_w^r) = 0$ degenerates into real and imaginary points. In all further considerations it will be assumed that not all of the zeros and poles of $f(z)$ are collinear.

where $x_1, x_2, \ldots, x_m$ are the zeros and poles that lie on that line, and where $\mu_1, \mu_2, \ldots, \mu_m$ are their multiplicities in $f(z)$.

Van den Berg's theorem follows from this when the function $f(z)$ is an integral rational function and $r = 1$, except that (as van den Berg and also Heawood failed to point out) the point of contact on a segment $z_i z_j$ does not divide the segment in the ratio of μ_i to μ_j in case some other zero of $f(z)$ lies on that same segment.

In case $\mu_i > r$, the curve $(\varphi_w^r) = 0$ will have a factor $\alpha_i^{\mu_i - r}$. We define $[\varphi_w^r] = 0$ to be that part of $(\varphi_w^r) = 0$ remaining after all the factors $\prod_{i=l+1}^{n} \alpha_i^{\mu_i - r}$ $(\mu_i > r,\ i = l+1, \ldots, n)$ of $(\varphi_w^r) = 0$ have been suppressed. The curve under investigation is $(\varphi_w^r) = 0$, but more particularly $[\varphi_w^r] = 0$. The curve in van den Berg's theorem is $[\varphi_w'] = 0$. It follows, from the discussion of $(F_w^r) = 0$,* that if (ν) and $[\nu]$ designate the degrees of $(\varphi_w^r) = 0$ and $[\varphi_w^r] = 0$ respectively,

$$(\nu) = r(k-1) + \sum_{i=k+1}^{n} \mu_i, \text{ and } [\nu] = r(k-1) + \sum_{i=k+1}^{n} \mu_i - \sum_{i=l+1}^{n} (\mu_i - r).$$

3. Points of contact on $z_i z_j$

Let $z_i z_j$ be two of the points $z_1, z_2, \ldots, z_n$ such that none of the remaining $n-2$ points lies on the line joining z_i to z_j. When this line $z_i z_j$ is tangent to $(\varphi_w^r) = 0$ and $[\varphi_w^r]$ we shall denote the multiplicity of its tangency by (ν_{ij}) and $[\nu_{ij}]$ respectively. Since the order of a multiple tangent is equal to the number of points of contact on it — distinct or coincident — we can construct the table given below.

It is clear, from this table, that the only case in which $(\varphi_w^r) = 0$ and $[\varphi_w^r] = 0$ are not tangent to $z_i z_j$ is that in which $\mu_i > 0$, $\mu_j > 0$, and $\mu_i + \mu_j \leqq r$. Furthermore $(\nu_{ij}) = [\nu_{ij}]$ except when $\mu_i > r$, or when $\mu_j > r$, or when both μ_i and μ_j are greater than r.

For $\mu_i > 0$ and $\mu_j > 0$ the points of contact of $[\varphi_w^r] = 0$ with the line $z_i z_j$ are always real and distinct and lie between z_i and z_j. The points of contact of $[\varphi_w^r] = 0$ separate the points of contact of $[\varphi_w^{r-1}] = 0$. If one of the μ's, say μ_i, is negative and if $\mu_i + \mu_j < 0$, the points of contact with $[\varphi_w^r] = 0$ are still finite, real, and distinct, but lie now on the extension of $z_i z_j$.† The points of contact of $[\varphi_w^r] = 0$ and of $[\varphi_w^{r-1}] = 0$ are mutually separated. If $\mu_i + \mu_j = 0$, the distribution of points of contact is the same as in the case $\mu_i + \mu_j < 0$, with one exception; namely, all but one of the points of con-

* See footnote on p. 243.

† A point is said to lie on the extension of $z_i z_j$ if z_j lies between the point and z_i.

TABLE 1.

	$\mu_i > r$			$0 \leqq \mu_i \leqq r$			$\mu_i < 0$		
	$(\nu_{ij}) \equiv \mu_i + \mu_j - r$	$[\nu_{ij}] \equiv r$		$(\nu_{ij}) \equiv \mu_i + \mu_j - r$	$[\nu_{ij}] \equiv \mu_i$		$(\nu_{ij}) \equiv \mu_j$	$[\nu_{ij}] \equiv r$	
$\mu_j > r$									
$r \geqq \mu_j > 0$				$(\nu_{ij}) \equiv \mu_i + \mu_j - r > 0$	$[\nu_{ij}] \equiv (\nu_{ij})$		$(\nu_{ij}) \equiv \mu_j$	$[\nu_{ij}] \equiv (\nu_{ij})$	
$\mu_j < 0$							$(\nu_{ij}) \equiv r$	$[\nu_j] \equiv (\nu_{ij})$	

tact are finite. The line through $z_i z_j$ is therefore an asymptote of $[\varphi_w^r] = 0$ with or without other finite points of contact on it, and, as the point of tangency at infinity counts for just one of the points of contact of that line with $[\varphi_w^r] = 0$, the two asymptotic branches of $[\varphi_w^r] = 0$ lie on different sides of $z_i z_j$. Furthermore, if m zeros are collinear and no pole is collinear with them, then the points of contact of $[\varphi_w^r] = 0$ with that line are still finite, real, and distinct. Moreover, if one pole is collinear with m zeros and the order of the pole is greater than the sum of the orders of the zeros, the points of contact with $[\varphi_w^r] = 0$ are still finite, real, and distinct. Finally, if $z_1, z_2, \ldots, z_m$ are collinear, so that $\mu_i > 0 (i = 1, \ldots, k')$, $0 < \mu_j \leqq r (j = k+1, \ldots, l')$, $\mu_\lambda > r$ $(\lambda = l'+1, \ldots, m)$, and if (ν_m) and $[\nu_m]$ designate the multiplicity of this tangent in $(\varphi_w^r) = 0$ and $[\varphi_w^r] = 0$ respectively, then

$$(8) \quad (\nu_m) = r(k'-1) + \sum_{i=k'+1}^{m} \mu_i,$$

$$\text{and} \quad [\nu_m] = (\nu) - \sum_{i=l'+1}^{m} (\mu_i - r).$$

4. THE TANGENTS FROM z_j TO $[\varphi_w^r] = 0$

Let the curve $\theta(u, v, w) = 0$ be defined by the identity

$$\varphi(u, v, w) \equiv \alpha_j^{\mu_j}\, \theta(u, v, w) \qquad (j \leqq n).$$

Then

$$\frac{\partial^r \varphi(u, v, w)}{\partial w^r} \equiv$$

$$\sum_{\lambda=0}^{r} \frac{r!}{(r-\lambda)!} \frac{\mu_j!}{(\mu_j-\lambda)!} \frac{\alpha_j^{\mu_j-\lambda}}{\lambda!} \frac{\partial^{r-\lambda} \theta(u, v, w)}{\partial w^{r-\lambda}},$$

and according as (1) $\mu_j \geqq r$, (2) $0 < \mu_j < r$, (3) $\mu_j < 0$, we have the following three expressions for $(\varphi_w^r) = 0$:

$$\text{(i)}\quad (\varphi_w^r) = \alpha^{\mu_j - r} \sum_{\lambda=0}^{r} A_\lambda\, \alpha^{r-\lambda}\, (\theta_w^{r-\lambda}) \prod_{i=1}^{k} \alpha_i^\lambda, \qquad A_\lambda = \frac{r!}{(r-\lambda)!}\,\frac{\mu_j!}{(\mu_j-\lambda)!}\,\frac{1}{\lambda},$$

$$\text{(ii)}\quad (\varphi_w^r) = \sum_{\lambda=0}^{\mu_j} A_\lambda\, \alpha^{\mu_j-\lambda}\, (\theta_w^{r-\lambda}) \prod_{i=1}^{k} \alpha_i^\lambda,$$

$$\text{(iii)}\quad (\varphi_w^r) = \sum_{\lambda=0}^{r} A_\lambda\, \alpha^{r-\lambda}\, (\theta^{r-\lambda}) \prod_{i=1}^{k} \alpha_i^\lambda / \alpha_j^\lambda.$$

Also in all three cases the identity $(\varphi_w^r) \equiv [\varphi_w^r] \cdot \prod_{i=l+1}^{n} \alpha_i^{\mu_i - r}$ holds.* It follows therefore that

$$\text{(i)}\quad [\varphi_w^r] = \alpha_j \sum_{\lambda=0}^{r-1} A_\lambda\, \Phi_\lambda + A_r \prod_{i=1}^{k} \alpha_i^r \cdot \prod_{i=k+1}^{l} \alpha_i^{\mu_i} \cdot \prod_{i=l+1,\, i\neq j}^{n} \alpha_i^r$$

$$\left(\Phi_\lambda \equiv \alpha_j^{r-\lambda-1}\, (\theta_w^{r-\lambda}) \prod_{\substack{i=1\\ i\neq j}}^{k} \alpha_i^\lambda \Big/ \prod_{\substack{i=l+1\\ i\neq j}}^{n} \alpha_i^{\mu_i - r}\right),$$

$$\text{(ii)}\quad [\varphi_w^r] = \alpha_j \sum_{\lambda=0}^{\mu_j - 1} A_\lambda\, \alpha^{\mu_j - r}\, \Phi_\lambda + A_{\mu_j} \prod_{i=1}^{k} \alpha_i^{\mu_j}$$

$$\cdot \prod_{i=k+1,\, \mu_i+\mu_j>r,\, i\neq j}^{l} \alpha_i^{\mu_i+\mu_j-r} \cdot \prod_{i=l+1}^{n} \alpha_i^{\mu_j} \cdot [\theta_w^{r-\mu_j}],\dagger$$

$$\text{(iii)}\quad [\varphi_w^r] = \alpha_j \sum_{\lambda=0}^{r-1} A_\lambda\, \Phi_\lambda + A_r \prod_{i=1,\, i\neq j}^{k} \alpha_i^r \cdot \prod_{i=k+1}^{l} \alpha_i^{\mu_i} \cdot \prod_{i=l+1}^{n} \alpha_i^r.$$

Hence we see again‡ that if $\mu_j \geq r$ or if $\mu_j < 0$, the totality of tangents from z_j to $[\varphi_w^r] = 0$ consists of the lines joining z_j to z_i $(i = 1, \ldots, k;\ i = l+1, \ldots, n;\ i \neq j)$, the order of the tangents in this case being r, and the lines

* The letter l designates the number of zeros whose order is less than r.

† We define $(\theta_w^{r-\mu_j}) \equiv [\theta_w^{r-\mu_j}] \prod_{i=k+1,\, \mu_i+\mu_j>r}^{n} \alpha^{\mu_i+\mu_j-r}$ by analogy with $[\varphi_w^r]$.

‡ We assume here that z_j is not collinear with any two of the remaining z's. In case of collinearity we use equation (8).

joining z_j to z_i $(i = k+1, \ldots, l)$, in which case the order of the tangents is μ_j. If, on the other hand, $0 < \mu_j \leqq r$, the totality of tangents from z_j to $[\varphi_w^r] = 0$ consists of (a) the lines joining z_j to z_i $(i = 1, \ldots, k;\ i = l+1, \ldots, n)$, the order of the tangents in this case being μ_j; (b) the lines joining z_j to z_i $(i = k+1, \ldots, l;\ i \neq j;\ \mu_i + \mu_j > r)$ in which case the order of the tangents is $\mu_i + \mu_j - r$; and (c) the lines from z_j tangent to $[\theta_w^{r-\mu_j}] = 0$.

In cases (a) and (b) above the points of contact of a tangent through $z_i z_j$ are given by Theorem 1. We proceed to determine the points of contact in case (c).

For the sake of simplicity let $\alpha_j = 0$ be the origin $(x_j = y_j = 0)$. Then for values of u, v, w which satisfy $\alpha_j = (\theta_w^{r-\mu_j}) = 0$, inasmuch as $(\theta^{r-\mu_j+1}) = \dfrac{\partial(\theta^{r-\mu_j})}{\partial w} \prod_{i=1}^{k} \alpha_i$, we have these three relations:

$$\frac{\partial(\varphi_w^r)}{\partial u} = \frac{r!}{(r-\mu_j)!} \frac{\partial(\theta^{r-\mu_j})}{\partial u} \prod_{i=1}^{k} \alpha_i^{\mu_j},$$

$$\frac{\partial(\varphi_w^r)}{\partial v} = \frac{r!}{(r-\mu_j)!} \frac{\partial(\theta_w^{r-\mu_j})}{\partial v} \prod_{i=1}^{k} \alpha_i^{\mu_j}, \quad \frac{\partial(\varphi_w^r)}{\partial w} = \frac{(r+1)\,r!}{(r-\mu_j+1)!} \cdot \prod_{i=1}^{k} \alpha_i^{\mu_j}$$

Now making use of the fact that the coördinates of the point of contact on a tangent (u', v', w') to the curve $\psi(u, v, w) = 0$ are given by

$$\frac{X}{\dfrac{\partial \psi}{\partial u'}} = \frac{Y}{\dfrac{\partial \psi}{\partial v'}} = \frac{1}{\dfrac{\partial \psi}{\partial w'}},$$

we get

$$X_{(\varphi_w^r)} = \frac{r-\mu_j+1}{r+1} \cdot X_{(\theta^{r-\mu_j})}, \quad \text{and} \quad Y_{(\varphi_w^r)} = \frac{r-\mu_j+1}{r+1} \cdot Y_{(\theta_w^{r-\mu_j})},$$

provided the tangent under consideration is not singular. Hence,

THEOREM 2. *When $\mu_j \geqq r$ or $\mu_j < 0$ $(j \leqq n)$, all the tangents from z_j to $[\varphi_w^r] = 0$ pass through the remaining zeros and poles of $f(z)$. When $0 < \mu_j < r$ $(j \leqq n)$, all the tangents from z_j either (a) pass through the poles of $f(z)$, or those zeros of $f(z)$ whose order added to μ_j is greater than r, or (b) are tangent to $[\theta_w^{r-\mu_j}] = 0$, where $\theta(u, v, w) \equiv \varphi(u, v, w)/\alpha_j^{\mu_j}$. The point of contact on a tangent to $[\theta_w^{r-\mu_j}] = 0$ divides the segment from z_j to the point of contact with $[\theta^{r-\mu_j}] = 0$ in the ratio of $r-\mu_j+1$ to μ_j, provided that the last tangent is not singular.*

While the equation of the tangents from a general point is not very illuminating, the equation of the tangents from any point at infinity is instructive. We consider this case next.*

5. Parallel tangents of $(\varphi_w^r) = 0$

THEOREM 3. *The tangents of $(\varphi_w^r) = 0$ parallel to any direction remain invariant if each of the points $z_1, z_2, \ldots, z_n$ is translated parallel to that direction an arbitrary distance.† If a line perpendicular to the direction of the tangents be taken for the X-axis and if $x_i \neq x_j$ $(i, j = 1, 2, \ldots, n)$, the equation of the vertical tangents is given by the rth derivative of*

$$\prod_{i=1}^{n} (x - x_i)^{\mu_i} = 0.$$

The proof of this theorem follows immediately from Corollary 1, Lemma 4. If the projection of a pole of $f(z)$ on the X-axis coincides with the projection of another pole, or with the projection of a root of the same function on the same axis, the vertical tangents of $(\varphi_w^r) = 0$ are given by the rth derivative of $\prod_{i=1}^{n} (x - x_i)^{\mu_i} = 0$ multiplied by a factor which is determined by Lemma 4. The factor does not add new distinct tangents, but merely increases the multiplicity of the old ones.

We can imagine the curve $(\varphi_w^r) = 0$ to be the envelope of its parallel tangents, and our investigation of its properties will, in large part, be based on a study of the distribution of its parallel tangents, and the distribution of the points of contact on the tangents through the zeros and poles of $f(z)$. Before we proceed to investigate the general curve $(\varphi_w^r) = 0$ we apply the above theorems to the curves $[\varphi_w^r] = 0$ of class two, i. e., to the conics.

6. Applications; $[r] = 2$

The degree of $[\varphi_w^r] = 0$ is two when $f(z)$ is of the type

(a) $f_1(z) = (z - z_1)^{\mu}(z - z_2)(z - z_3)$ $(\mu > 0, r = 1, \ldots, \mu; \mu < 0, r = 1, 2, \ldots)$,

(b) $f_2(z) = (z - z_1)^{\mu_1}(z - z_2)^{\mu_2}(z - z_3)^{\mu_3}$ $(r = 1)$,

(c) $f_3(z) = \prod_{i=1}^{n} (z - z_i)^{\mu_i}$ $\left(\mu_i > 0, r = \sum_{i=1}^{n} \mu_i - 2\right)$.

* The writer wishes to express his obligation to Professor W. C. Graustein, of Harvard University, for certain valuable suggestions he has made concerning the subject matter treated in the two preceding sections.

† Provided $\theta'(x) \not\equiv 0$; see Lemma 4.

We propose to study each of those cases in detail and to show how the foci of $[\varphi_i^r] = 0$* can be constructed by means of ruler and compass.

TYPE (a) $f(z) = (z - z_1)^\mu (z - z_2)(z - z_3)$. The point of contact of $[\varphi_1^r] = 0$ with $z_1 z_2$ divides the segment $z_1 z_2$ in the ratio of $\mu - r + 1$ to r, internally or externally according as $\mu \gtrless 0$. Similarly the point of contact with $z_1 z_3$ divides the segment $z_1 z_3$ in the same ratio. Let B and C be the points of contact on $z_1 z_2$ and $z_1 z_3$ respectively; then BC is parallel to $z_2 z_3$ and therefore the center of the conic $[\varphi_1^r] = 0$ lies on that median of the triangle $z_1 z_2 z_3$ which is drawn from z_1. The tangent parallel to $z_1 z_3$ intersects $z_1 z_2$ in a point which divides $z_1 z_2$ in the ratio of $\mu - r + 2$ to r, internally or externally according as $\mu \gtrless 0$. Also the tangent parallel to $z_1 z_2$ divides $z_1 z_3$ in the same ratio. Hence the center of the conic divides the median in the ratio of $\mu - r + 2$ to r. Furthermore, when $\mu > 0$ the conic is always an ellipse; when $\mu < 0$, but $\mu + 1 \neq 0$ the conic is an ellipse, a parabola, or a hyperbola according as $\mu + 2 \gtreqless 0$; and finally, when $\mu + 1 = 0$ the conic is a hyperbola, for $r = 1$, but degenerates into two pencils of lines parallel to $z_1 z_2$ and $z_1 z_3$ for all other values of r. These results follow at once from the fact that a conic is an ellipse or a hyperbola according as it passes through a point between or not between two parallel tangents, while it is a parabola if it touches the line at infinity.

To construct the foci of $[\varphi_1^r] = 0$ by means of a ruler and a compass we make use of the two following theorems:† The conjugate diameters of a central conic set up on a circle through the center of the conic an involution whose rectangular pair are the axes of the conic; and a tangent and a normal at any point of a central conic intersect the minor axis of the conic in points which are concyclic with the foci.‡ Hence we have the construction: From O, the center of $[\varphi_1^r] = 0$, draw OE and OF parallel to $z_1 B$ and $z_1 C$ respectively. Then OB, OE and OC, OF are two pairs of conjugate directions of the last conic. Now take any point O' as a center and draw a circle through O. Let this circle intersect OB, OE, OC, and OF in B', E', C' and F' respectively. Denote by D the intersection of $B'E'$ and $C'F'$, and let OD intersect the circle in the points M and N. Then OM and ON are in the direction of the axes of $[\varphi_1^r] = 0$. Let $z_1 B$ and the perpendicular to $z_1 B$ at B intersect the minor axis§ in P and Q respectively. Then the circle on PQ as a diameter intersects the major axis in the foci of $[\varphi_1^r] = 0$.‖

* $[\varphi_i^r] = 0$ is the φ curve corresponding to $f_i(z)$ $\quad (i = 1, 2, 3)$.

† This method of constructing the axes of a conic was called to my attention by Professor C. M. Sparrow, of the University of Virginia.

‡ Casey, *Analytical Geometry*, 2d edition, p. 215.

§ The normal at any point (not a vertex) of a central conic first intersects the major axis and then the minor axis.

‖ In the case of a cubic with simple roots compare the above method with one given by J. L. Walsh, Annals of Mathematics, vol. 22 (1920), p. 142.

The construction is much simplified if the triangle $z_1 z_2 z_3$ is isosceles, $z_1 z_2 = z_1 z_3$. Then the median through z_1 and the perpendicular to it at O are the axes of $[\varphi_1^r] = 0$, and, as before, $z_1 B$ and the normal at B determine a circle which intersects the major axis in the foci. If we take the above median for the axis of reals, the roots of $f_1^r(z) = 0$ are real or imaginary according as the intersection of the normal at B and the axis of reals lies inside or outside the segment $z_1 O$. Furthermore, if the normal at B passes through O, the roots of $f_1^r(z) = 0$ are coincident and $[\varphi_1^r] = 0$ is a circle.

In case $[\varphi_1^r] = 0$ is a parabola ($\mu = 2$), the construction is also quite simple. We have merely to make use of the fact that the tangent at any point of a parabola bisects the angle between the line joining that point to the focus and the perpendicular to the directrix from that point. In our case, of course, the directrix is perpendicular to the median through z_1. Hence we have the construction: Draw the perpendicular bisectors of $z_1 B$ and $z_1 C$, and let them intersect the median through z_1 in the points E and F respectively. Then the intersection of BE and CF is the desired focus.*

TYPE (b) $f_2(z) = (z-z_1)^{\mu_1}(z-z_2)^{\mu_2}(z-z_3)^{\mu_3}$. When the powers ($\mu$) are all positive or all negative, the conic $[\varphi_2'] = 0$ is an ellipse. If one of the powers is negative it is an ellipse, parabola, or hyperbola according as $\mu_1+\mu_2+\mu_3 \lesseqgtr 0$. If two of the powers are negative it is an ellipse, a parabola, or a hyperbola according as $\mu_1+\mu_2+\mu_3 \gtreqless 0$. When the conic is central its center is the center of gravity of the three masses (negative or positive) $\mu_2+\mu_3$, $\mu_3+\mu_1$, $\mu_1+\mu_2$ considered located at the vertices of the triangle $z_1 z_2 z_3$ in the order designated. If the conic is a parabola, the signs of all three powers can not be the same. Let μ_2 and μ_3 be of like sign. Then the line joining z_1 to the midpoint of the join of the point of contact on $z_1 z_2$ and the point of contact on $z_1 z_3$ is perpendicular to the directrix of the parabola. Having the center of a central conic or the direction of the directrix of a parabola in addition to the points of contact on two tangents, we can construct the foci or focus by means of ruler and compass as in case (a).

Again the construction of the foci may be considerably simplified for particular functions $f_2(z)$. We mention only two such cases here. If $\mu_1+\mu_2 = 0$, the side $z_1 z_2$ is an asymptote, and the center divides the side $z_1 z_2$ in the ratio of $\mu_2+\mu_3$ to $\mu_1+\mu_3$. The fourth harmonic of OC, OF and Oz_1, where O is the center, C the point of contact on $z_1 z_3$, and OF a line parallel to $z_1 z_3$, is the second asymptote. If also $\mu_1+\mu_3 = 0$, then $z_1 z_3$ is the second asymptote. In either case, the bisectors of the angles between the asymptotes are the axes, and the tangent $z_2 z_3$, together with the normal at

* It follows therefore that the focus of $[\varphi_1^r] = 0$ lies on the symmedian through z_1 for all values of r, and that the distance between two successive foci is constant.

the point of contact on it, determines a circle which intersects the major axis in the foci.

TYPE (c) $f_3(z) = \prod_{i=1}^{n} (z - z_i)^{\mu_i}$. By means of a repeated application of Theorem 2 we can construct the tangents from z_i to $[\varphi_3^r] = 0$ with their points of contact. The remaining part of the construction of the foci of $[\varphi_3^r] = 0$ is the same as that of $[\varphi_1^r] = 0$.*

7. THE POSITION OF THE CURVE $[\varphi_w^r] = 0$ WITH RESPECT TO THE ZEROS AND POLES OF $f(z)$

Case 1. $f(z)$ is a rational integral function of z.

When $r = 1$ we have

$$[\varphi_w^1] = \sum_{j=1}^{n} \mu_j \frac{\partial \prod_{i=1}^{n} \alpha_i}{\partial \alpha_j} = 0,$$

whose vertical tangents are given by

$$\sum_{j=1}^{n} \mu_j \frac{\partial \prod_{i=1}^{n} (x - x_i)}{\partial (x - x_j)} = 0.$$

It follows from Rolle's theorem on the roots of the derivative of a polynomial, that, if $x_i \neq x_j$ $(i, j = 1, \ldots, n)$, there is one and only one root of $f'(z)$ between any two successive vertical tangents of $[\varphi_w^r] = 0$, and that there are always two roots such that no tangent lies outside of them. If in projecting the roots on some line — say the X-axis — it should happen that x_i and x_j coincide, the perpendicular to the line at x_i is a tangent to $[\varphi_w^1] = 0$. Moreover, if the projections of m zeros coincide, the perpendicular at that point is a tangent of order $m - 1$ with $m - 1$ finite and distinct points of contact on it. If no three roots of $f(z)$ are collinear, the curve $[\varphi_w^1] = 0$ has always $n - 1$ finite and distinct tangents parallel to any direction and therefore has no singularities whatsoever.† On the other hand, if three or more

* For lack of space all details of this construction are omitted. However, it should be noticed that in case all the z's of $f_i(z)$ are collinear the displacement of one of the z's vertically and the construction of the vertical tangents of the $[\varphi_i^r] = 0$ corresponding to the new configuration of the z's gives the roots of $f_i'(z) = 0$ by Theorem 3.

† By singularities in a class curve we mean multiple tangents, or inflexions, or combinations of these.

of the roots of $f(z)$ are collinear, the curve $[\varphi_w^1] = 0$ has multiple tangents, but still has no inflexions, for the points of contact on such a multiple tangent are finite and distinct. Nor has it any asymptotes, for the number of tangents parallel to an asymptotic direction cannot exceed $n - 2$ in a curve of class $n - 1$, since the point at infinity is on the curve. It has no infinite branches, for all its tangents are at a finite distance from the origin.

The extension to $[\varphi_w^r] = 0$ follows without difficulty. There is one and only one tangent of $[\varphi_w^{r-1}] = 0$ parallel to and between any two successive parallel tangents of $[\varphi_w^r] = 0$, and there are two tangents of $[\varphi_w^{r-1}] = 0$ such that no tangent of $[\varphi_w^r] = 0$ lies outside of them. If no three zeros of $f(z)$ are collinear, $[\varphi_w^r] = 0$ has always

$$\sum_{i=1}^{n} \mu_i - r - \sum_{\substack{i=1 \\ \mu_i > r}}^{n} (\mu_i - r)$$

finite and distinct tangents parallel to any direction, and therefore has no singularities whatsoever. Furthermore, if three or more of the zeros of $f(z)$ are collinear, $[\varphi_w^r] = 0$ may have multiple tangents, but still can have no inflexions. Finally, $[\varphi_w^r] = 0$ has no asymptotes and does not touch the line at infinity. Hence $[\varphi_w^r] = 0$ is a closed curve.

If the tangents to $[\varphi_w^r] = 0$ from some point are all real, and if this point be considered to vary, then the tangents from it to $[\varphi_w^r] = 0$ will remain real as long as it does not cross the curve. For the curve $[\varphi_w^r] = 0$ has no asymptotes and no inflexions. Moreover, as this point crosses the curve it gains two tangents or loses two tangents according as it crosses from the concave or convex side of the curve. But as all the tangents from this point are real it can not gain any tangents; and, therefore, in crossing the curve, it meets the convex side of the curve first. Furthermore, from the point at infinity all the tangents are real. Consequently all the tangents to the curve $[\varphi_w^r] = 0$ drawn from any point outside of it are real, and hence, not only is $[\varphi_w^r] = 0$ closed, but its external contour is convex toward all points outside of it.

THEOREM 4. *The curve* $[\varphi_w^r] = 0$ *is closed, has no inflexions, and has no multiple tangents except perhaps through collinear zeros of* $f(z)$.* *It lies between the extreme tangents parallel to any direction, and it also lies between the extreme tangents drawn from any one of the zeros of* $f(z)$ *which lies on the least*

* To determine whether or not a line through collinear zeros is a multiple tangent use equation (8).

rectilinear polygon containing all the zeros of $f(z)$. It is convex toward all zeros of $f(z)$ whose order is not less than r.*

Case 2. $f(z)$ is a rational fractional function of z.

If $f(z)$ has one single pole whose order is greater than the sum of the orders of the zeros, all the tangents of $[\varphi_w^r] = 0$ parallel to any direction are still real. Furthermore, $[\varphi_w^r] = 0$ still has no multiple tangents except perhaps through the pole and through a zero or perhaps through two or more collinear zeros. The points of contact on such a multiple tangent are always finite, real, and distinct. Hence $[\varphi_w^r] = 0$ has no inflexions. Continuing this reasoning as we did in the case when $f(z)$ was a rational function, we finally conclude that $[\varphi_w^r] = 0$ is closed, that its external contour is convex toward all points lying outside of it, and that it lies between the extreme tangents parallel to any direction.†

When $f(z)$ has more than one pole or when the order of the single pole is not greater than the sum of the orders of its zeros, these conclusions do not generally hold. Even then we can usually determine pairs of slopes such that, for all tangents whose slopes lie between these limiting values, most of the conditions mentioned above will hold.

8. Generalization of the Gauss'-polygon theorem

We are now in a position to generalize the Gauss'-polygon theorem very simply.

The foci of a curve are the intersections of the tangents to the curve from the circular points at infinity. Consequently, at least two of the tangents from a focus are imaginary. Therefore the foci of $[\varphi_w^r] = 0$, when $f(z)$ is a rational integral function, lie between the extreme tangents of $[\varphi_w^r] = 0$ parallel to any direction. Hence,

THEOREM 5. *If the x-intercepts of lines through the zeros of a rational integral function, $f(z)$, parallel to any direction, be considered as the zeros of a new real integral function, then those zeros of the rth derivative of $f(z)$ that are distinct from the zeros of $f(z)$ lie between the two lines parallel to the same direction and passing through the least and greatest zero of the rth derivative of the new rational integral function.*‡

* By considering tangents parallel to this polygon we conclude that $[\varphi_w^r] = 0$ lies inside this polygon.

† Also the tangents from any point outside of it are all real.

‡ This theorem may also be stated in the form suggested by Professor Kempner, of the University of Illinois: Let the zeros of a polynomial $f(z)$ be $x_1 + iy_1, \ldots, x_n + iy_n$, and let the zeros of $f^r(z)$ be $x_1' + iy_1', \ldots, x_m' + iy_m'$, where $x_1' \leqq x_2' \leqq \cdots \leqq x_m'$. If now $\bar{x}_1$ and $\bar{x}_m$ be the least and greatest (real) root of the rth derivative of $g(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n)$, then $\bar{x}_1 < x_1'$ and $\bar{x}_m > x_m'$.

The Gauss'-polygon theorem follows from this when $r = 1$ and the directions in the theorem are taken parallel to the sides of the least rectilinear polygon containing all the roots of $f(z)$.

Finally, Theorem 5 applies not only to a rational integral function as stated, but to any rational fractional function whose corresponding curve $[\varphi_w^r] = 0$ is closed, has no inflexions, and all of whose tangents from any point at infinity are real.* The theorem is therefore applicable to a rational function of one multiple pole whose order is greater than the sum of the orders of the zeros.†

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* Cf. Bôcher, Proceedings of the American Academy of Arts and Sciences, vol. 40 (1904), p. 478. Also Walsh, these Transactions, vol. 19 (1918), p. 297.

† Part II of this paper, which deals primarily with the tracing of the φ-curves, will appear separately.

ORTHOGONAL SYSTEMS OF HYPERSURFACES IN A GENERAL RIEMANN SPACE*

BY

LUTHER PFAHLER EISENHART

1. **Introduction.** Consider a general Riemann space of n dimensions and write its fundamental quadratic form thus,

$$ds^2 = g_{ij}\,dx^i\,dx^j \qquad (g_{ij} = g_{ji}).\dagger \tag{1}$$

If a_{ij} are the components of any other covariant symmetric tensor of the second order, and ϱ_h is a root of the equation

$$|a_{rs} + \varrho g_{rs}| = 0, \tag{2}$$

the functions λ_h^r $(r = 1, \ldots, n)$, defined by

$$(a_{rs} + \varrho_h g_{rs})\,\lambda_h^r = 0, \tag{3}$$

are the contravariant components of a vector in the space.

By hypothesis the form (1) is positive definite and consequently the roots of (2) are real. If all the roots are simple, equations (3) define uniquely n directions at a point. Thus if λ_h^r and λ_k^r are the components of the vectors corresponding to distinct roots ϱ_h and ϱ_k, it follows from (3) that

$$g_{rs}\,\lambda_h^r\,\lambda_k^s = 0, \tag{4}$$

that is, the corresponding directions are orthogonal. Moreover, if ϱ_h is a multiple root of order m,‡ equations (3) admit solutions linearly expressible in terms of m solutions.§ If we take m of these directions which are mutually orthogonal, and proceed in like manner with every multiple root, we have at

* Presented to the Society, April 28, 1923.

† Here i and j are summed from 1 to n in accordance with the convention now generally in use. This convention will be used throughout this paper.

‡ We exclude the case where $m = n$, that is, when $a_{rs} + \rho g_{rs} = 0$.

§ Weierstrass, Monatsberichte der Akademie zu Berlin, 1858, p. 207.

each point of space n directions mutually orthogonal. They are the principal directions determined by the tensor α_{rs}. These directions determine n congruences of curves in the space, any congruence being defined analytically by

$$\frac{dx^1}{\lambda_h^1} = \cdots = \frac{dx^n}{\lambda_h^n}. \tag{5}$$

It is the purpose of this paper to determine conditions necessary and sufficient that the curves of each congruence be orthogonal to a family of hypersurfaces, V_{n-1}. In this case, there pass through a curve C_h, of the congruence defined by (5), $n-1$ hypersurfaces respectively orthogonal to the congruences $C_k (k = 1, \ldots, n; k \neq h)$. When these hypersurfaces are taken as parametric, the functions $g_{ij} (i \neq j)$ in (1) are equal to zero, and the space possesses an n-uple orthogonal system of hypersurfaces. In this case we say that the n-uple of congruences is *normal*.

When the roots of (2) are simple or double the conditions are readily obtained, but for roots of third and higher order the conditions are quite involved. However, in each case the problem is reduced to an algebraic one.

2. **General equations.** Without loss of generality, we assume that the functions λ_h^r are chosen so that

$$g_{rs} \lambda_h^r \lambda_h^s = 1. \tag{6}$$

As given by (3) these functions are the contravariant components of the direction. Their covariant components are given by

$$\lambda_{h,r} = g_{rp} \lambda_h^p. \tag{7}$$

If $\lambda_{h,r/s}$ denote the components of the tensor which is the covariant derivative of $\lambda_{h,r}$ with respect to (1), the functions γ_{hij}, defined by

$$\gamma_{hij} = \lambda_i^r \lambda_j^s \lambda_{h,r/s}, \tag{8}$$

are invariants; they are called rotations by Ricci and Levi-Civita.*

From (8) it follows that

$$\lambda_{h,r/s} = \sum_{i,j}^{1 \ldots n} \gamma_{hij} \lambda_{i,r} \lambda_{j,s}. \tag{9}$$

* Mathematische Annalen, vol. 54 (1901), p. 148; Wright, *Invariants of Quadratic Differential Forms*, Cambridge Tract, No. 9, p. 68.

If we denote by g^{ij} the cofactor of g_{ij} in the determinant

$$g = |g_{ij}| \tag{10}$$

divided by g, we may replace (7) by

$$\lambda_h^p = g^{pr}\,\lambda_{h,r}. \tag{11}$$

Taking the covariant derivative of (11) and making use of (9), we have

$$\lambda_{h/s}^p = \sum_{i,j}^{1\ldots n} \gamma_{hij}\,\lambda_i^p\,\lambda_{j,s}, \tag{12}$$

since the covariant derivatives of g_{ij} and g^{ij} are zero. Hence

$$\gamma_{hij} = \lambda_{i,p}\,\lambda_j^s\,\lambda_{h/s}^p. \tag{13}$$

From (4), (6) and (11) we have

$$\lambda_h^p\,\lambda_{k,p} = \delta_{hk}, \tag{14}$$

where

$$\delta_{hk} = 1 \text{ for } h = k, \text{ and } 0 \text{ for } h \neq k. \tag{15}$$

Differentiating (14) with respect to x^s, multiplying by λ_l^s and summing for s, we have, in consequence of (9), (12) and (14),

$$\gamma_{hkl} + \gamma_{khl} = 0 \qquad (h, k, l = 1, \ldots, n), \tag{16}$$

and consequently

$$\gamma_{hhl} = 0 \qquad (h, l = 1, \ldots, n). \tag{17}$$

Ricci* has shown that the conditions of integrability of equations (9) may be written in the form

$$\gamma_{hi,kl} = B_{qr,st}\,\lambda_h^q\,\lambda_i^r\,\lambda_k^s\,\lambda_l^t, \tag{18}$$

* Loc. cit., p. 157; Wright, p. 76.

where $B_{qr,st}$ are the components of the Riemann tensor, that is

$$B_{qr,st} = g_{rp}\left(\frac{\partial \Gamma^p_{qs}}{\partial x^t} - \frac{\partial \Gamma^p_{qt}}{\partial x^s} + \Gamma^p_{at}\,\Gamma^a_{qs} - \Gamma^p_{as}\,\Gamma^a_{qt}\right),$$

(19)

$$\Gamma^p_{qs} = \left\{\begin{matrix} qs \\ p \end{matrix}\right\} = \tfrac{1}{2} g^{pa}\left(\frac{\partial g_{qa}}{\partial x^s} + \frac{\partial g_{sa}}{\partial x^q} - \frac{\partial g_{qs}}{\partial x^a}\right),$$

and by definition

$$\gamma_{hi,kl} = \frac{\partial \gamma_{hik}}{\partial s_l} - \frac{\partial \gamma_{hil}}{\partial s_k} + \sum_j^{1\ldots n}\left[\gamma_{hij}\,(\gamma_{jkl} - \gamma_{jlk}) + \gamma_{jhl}\,\gamma_{jik} - \gamma_{jhk}\,\gamma_{jil}\right], \tag{20}$$

it being understood that for any function f the expression $\partial f/\partial s_l$ means

$$\frac{\partial f}{\partial s_l} = \frac{\partial f}{\partial x^m}\lambda^m_l. \tag{21}$$

From (18), (19) and (20) it follows that

$$\gamma_{hi,kl} = -\gamma_{ih,kl} = -\gamma_{hi,lk} = \gamma_{kl,hi}. \tag{22}$$

3. **Simple roots.** The results of the preceding section hold for any orthogonal n-uple. We apply them in this section to the n-uple determined by (3). Since (4) is satisfied, whether the functions λ^r_h and λ^s_k correspond to different simple roots of (2), or to the same multiple root if such exist, we have from (3)

$$a_{rs}\,\lambda^r_h\,\lambda^s_k = 0 \qquad (h,\ k = 1,\ \ldots,\ n;\ h,\ k \neq). \tag{23}$$

Also from (3) we have

$$a_{rs}\,\lambda^r_h\,\lambda^s_h = -\varrho_h, \tag{24}$$

in consequence of (6).

If we differentiate (23) with respect to x^t, we get, in consequence of (12),

$$a_{rs/t}\,\lambda^r_h\,\lambda^s_k + a_{rs}\sum_{i,j}^{1\ldots n}(\lambda^s_k\,\gamma_{hij}\,\lambda^r_i\,\lambda_{j,t} + \lambda^r_h\,\gamma_{kij}\,\lambda^s_i\,\lambda_{j,t}) = 0, \tag{25}$$

where $a_{rs/t}$ is the covariant derivative of a_{rs} with respect to x^t. Because of (23), (24) and (16) this is reducible to

$$a_{rs/t}\,\lambda^r_h\,\lambda^s_k + \sum_j^{1\ldots n}(\varrho_h - \varrho_k)\,\gamma_{hkj}\,\lambda_{j,t} = 0. \tag{26}$$

Multiplying by λ_l^t and summing for t, we get

$$\alpha_{rs/t}\,\lambda_h^r\,\lambda_k^s\,\lambda_l^t+(\varrho_h-\varrho_k)\,\gamma_{hkl}=0 \qquad (h\neq k). \tag{27}$$

Ricci* has shown that a necessary and sufficient condition that an orthogonal *n-uple* be normal is that

$$\gamma_{hkl}=0 \qquad (h,k,l=1,\ldots,n;\ h,k,l\neq). \tag{28}$$

Hence from (27) we have the theorem:

If all the roots of equation (2) *are simple, a necessary and sufficient condition that the n-uple defined by* (3) *be normal is that the functions* λ_h^r *satisfy the conditions*

$$\alpha_{rs/t}\,\lambda_h^r\,\lambda_k^s\,\lambda_l^t=0 \qquad (h,k,l=1,\ldots,n;\ h,k,l\neq). \tag{29}$$

From (20) and (28) it follows that for a normal n-uple

$$\gamma_{hi,kl}=0 \qquad (h,i,k,l\neq), \tag{30}$$

and consequently the functions λ_h^r of a normal n-uple satisfy

$$B_{qr,st}\,\lambda_h^q\,\lambda_i^r\,\lambda_k^s\,\lambda_l^t=0 \qquad (h,i,k,l\neq). \tag{31}$$

4. **Double roots. Normal congruences in 3-space.** We consider next the case when one of the roots of equation (2) is of order m. We denote by λ_i^r $(i=1,\ldots,m)$ the contravariant components of m mutually orthogonal directions corresponding to this root and by λ_e^r $(e=m+1,\ldots,n)$ the components corresponding to the other roots, it being understood that if there are one, or more, other multiple roots, the congruences corresponding to this root be taken orthogonal to one another. Consequently (4) and (6) hold.

The contravariant components $\lambda_i'^r$ $(i=1,\ldots,m)$ of any other orthogonal congruences corresponding to the multiple root of order m are given by

$$\lambda_i'^r=\sum_a^{1\ldots m} t_i^a\,\lambda_a^r \qquad (i=1,\ldots,m;\ r=1,\ldots,n), \tag{32}$$

* Loc. cit., p. 151; Wright, p. 70.

where the t's are any functions of the x's subject to the conditions

$$\sum_{\alpha}^{1\ldots m} t_i^{\alpha}\, t_j^{\alpha} = \delta_{ij}, \tag{33}$$

the δ's being defined by (15). We denote by γ'_{hke} the rotations for this set of vectors, and seek under what conditions a set of functions t can be found so that

$$\gamma'_{hij} = 0 \qquad (h = 1, \ldots, m;\ i, j = 1, \ldots, n;\ h, i, j \neq). \tag{34}$$

For $i, j = m+1, \ldots, n\ (i \neq j)$ we have from (13) and (32)

$$\gamma'_{hij} = \lambda_{i,p}\, \lambda_j^s \sum_{\alpha}^{1\ldots m} \left(t_h^{\alpha}\, \lambda_{\alpha/s}^{p} + \lambda_{\alpha}^{p} \frac{\partial}{\partial x^s} t_h^{\alpha} \right) = \sum_{\alpha}^{1\ldots m} t_h^{\alpha}\, \gamma_{\alpha ij}.$$

Hence we must have $\sum_{\alpha}^{1\ldots m} t_h^{\alpha}\, \gamma_{\alpha ij} = 0$. Since these equations must be satisfied by $h = 1, \ldots, m$, we must have $\gamma_{hij} = 0$.

For $i = m+1, \ldots, n;\ j = 1, \ldots, m$ we must have in like manner

$$\sum_{\alpha,\beta}^{1\ldots m} t_h^{\alpha}\, t_j^{\beta}\, \gamma_{\alpha i\beta} = 0 \quad (h, j = 1, \ldots, m,\ h \neq j;\ i = m+1, \ldots, n). \tag{35}$$

Holding j fixed and varying h, we get in consequence of (33)

$$\frac{\sum_{\beta}^{1\ldots m} t_j^{\beta}\, \gamma_{1i\beta}}{t_j^1} = \cdots = \frac{\sum_{\beta}^{1\ldots m} t_j^{\beta}\, \gamma_{mi\beta}}{t_j^m} = \varrho.$$

From these equations and the above results we find that we must have

$$\gamma_{hij} = 0 \quad (h = 1, \ldots, m;\ i = m+1, \ldots, n;\ j = 1, \ldots, n;\ h, i, j \neq), \tag{36}$$

and

$$\gamma_{eaa} = \gamma_{eii} \qquad (e = m+1, \ldots, n;\ \alpha, i = 1, \ldots, m). \tag{37}$$

In consequence of (16) we have from (36) also

$$\gamma_{hij} = 0 \quad (h = m+1, \ldots, n;\ i = 1, \ldots, m;\ j = 1, \ldots, n;\ h, i, j \neq). \tag{38}$$

From the foregoing discussion it follows conversely that if (36), (37) and (38) are satisfied by any set of mutually orthogonal congruences corresponding to a root of order m, they are satisfied by every such set. From (27) it follows that (37) is equivalent to

$$(39) \qquad \alpha_{rslt}\,\lambda_e^r(\lambda_\alpha^s\,\lambda_\alpha^t - \lambda_i^s\,\lambda_i^t) = 0 \quad (e = m+1, \ldots, n;\ \alpha, i = 1, \ldots, m).$$

Consider next the case where (34) are satisfied when $h, i = 1, \ldots, m$; $j = m+1, \ldots, n$; $i \neq j$. Now

$$\gamma'_{hij} = \sum_{\alpha,\beta}^{1\ldots m} t_i^\beta \left[t_h^\alpha\,\gamma_{\alpha\beta j} + \frac{\partial}{\partial s_j}(t_h^\beta)\right].$$

In consequence of (16) and (33) we have identically

$$\sum_{\alpha,\beta}^{1\ldots m} t_h^\beta \left[t_h^\alpha\,\gamma_{\alpha\beta j} + \frac{\partial}{\partial s_j}(t_h^\beta)\right] = 0.$$

From this equation and those obtained by equating to zero the right-hand member of the above equation, we have

$$(40) \qquad \frac{\partial}{\partial s_j}(t_h^\beta) = \sum_{\alpha}^{1\ldots m} t_h^\alpha\,\gamma_{\beta\alpha j} \quad (\beta, h = 1, \ldots, m;\ j = m+1, \ldots, n).$$

Suppose that the given space is of the third order; that equation (2) admits a double root; that equations (29) and (39) are satisfied and that λ_3^r $(r = 1,2,3)$ are the components of the vector corresponding to the simple root of (2). By a transformation of coördinates we can take the surfaces orthogonal to this congruence for the surfaces $x_3 = \text{const}$. Then $\lambda_3^1 = \lambda_3^2 = 0$, $\lambda_3^3 = 1/\sqrt{g_{33}}$. If we put

$$t_1^1 = \cos\theta, \quad t_1^2 = \sin\theta, \quad t_2^1 = -\sin\theta, \quad t_2^2 = \cos\theta,$$

equations (33) are satisfied, and in place of (40) we have the single equation

$$\frac{\partial\,\theta}{\partial\,x^3} = \sqrt{g_{33}}\,\gamma_{213}$$

for the determination of θ, and consequently of the two normal congruences corresponding to the double root. Hence:

A necessary and sufficient condition that the equations (3) *for a* 3*-space, for which* (2) *admits a double root, lead to a triple of normal congruences is that equations* (29) *and* (39) *be satisfied by the direction corresponding to the simple root and by any set of mutually orthogonal directions corresponding to the double root.*

If the space is of higher order than the third, the conditions of integrability of (40) must be considered. These conditions are expressible in the form*

$$(41) \qquad \left(\frac{\partial}{\partial s_k}\frac{\partial}{\partial s_j} - \frac{\partial}{\partial s_j}\frac{\partial}{\partial s_k}\right) f + \sum_r^{1\ldots n} (\gamma_{rjk} - \gamma_{rkj}) \frac{\partial f}{\partial s_r} = 0.$$

Applying this condition to equations (40), we obtain, in consequence of (36), (38) and (20),

$$(42) \qquad \sum_\alpha^{1\ldots m} l_h^\alpha \gamma_{\beta\alpha,jk} = 0.$$

Since equations of this form must be hold for $h = 1, \ldots, m$, we must have

$$(43) \qquad \gamma_{\alpha\beta,jk} = 0 \qquad (\alpha, \beta = 1, \ldots, m; j, k = m+1, \ldots, n).$$

We have seen in (30) that this condition must be satisfied by the functions γ'_{hij} of a normal congruence. From (32) we have

$$(44) \qquad \gamma'_{\alpha\beta,jk} = B_{qr,st} \lambda_\alpha'^q \lambda_\beta'^r \lambda_j^s \lambda_k^t = \sum_{a,b}^{1\ldots m} \gamma_{ab,jk} l_\alpha^a l_\beta^b.$$

Hence if (43) is satisfied by any set of congruences corresponding to a multiple root, it is satisfied by every such set.

The conditions (36), (37), (38) and (40) are the only ones applying to a double root of equation (2). Hence:

If the roots of equation (2) *are simple and double at most, a necessary and sufficient condition that there exist a normal n-uple whose components satisfy* (3)

* Ricci and Levi-Civita, p. 150; Wright, p. 69.

is that any orthogonal n-uple satisfying (3) *shall satisfy* (29), *in which h and k do not refer to the same root of* (2), (39), *and also the equations*

$$B_{qr,st}\,\lambda_\alpha^q\,\lambda_\beta^r\,\lambda_j^s\,\lambda_k^t = 0 \qquad (\alpha, \beta, j, k \neq), \tag{45}$$

in which α and β refer to the same double root, and j and k to any other root or roots.

5. **Multiple roots of third and higher orders.** If $m > 2$, we must consider also the case of equations (34) for $h, i, j = 1, \ldots, m;\ h, i, j \neq$. Now we must have

$$\sum_{\alpha,\beta,\sigma}^{1\ldots m} t_i^\beta\, t_j^\alpha \left[t_h^\sigma\, \gamma_{\sigma\beta\alpha} + \frac{\partial}{\partial s_\alpha}\, t_h^\beta \right] = 0 \qquad (h, i, j = 1, \ldots, m;\ h, i, j \neq). \tag{46}$$

Since this equation is satisfied identically for $i = h$, equations (46) hold for $i = 1, \ldots, m$, but $i \neq j$, and consequently (46) may be replaced by

$$\sum_{\alpha,\sigma}^{1\ldots m} t_j^\alpha \left[t_h^\sigma\, \gamma_{\sigma\beta\alpha} + \frac{\partial}{\partial s_\alpha}\, t_h^\beta - A_h\, \delta_{\alpha\beta} \right] = 0,$$

where A_h is to be determined. Since this equation must be satisfied by all values of j except h, it may be replaced by

$$\frac{\partial}{\partial s_\alpha}\, t_h^\beta + \sum_{\sigma}^{1\ldots m} t_h^\sigma\, \gamma_{\sigma\beta\alpha} - A_h\, \delta_{\alpha\beta} = B_\beta\, t_h^\alpha,$$

where B_β is to be determined. Multiplying this equation by t_h^β and summing for β, we find that $A_h = -\sum_{\beta}^{1\ldots m} B_\beta\, t_h^\beta$, and the above equation becomes

$$\frac{\partial}{\partial s_\alpha}\, t_h^\beta = \sum_{\sigma}^{1\ldots m} t_h^\sigma\, (\gamma_{\beta\sigma\alpha} - \delta_{\alpha\beta}\, B_\sigma + B_\beta\, \delta_{\sigma\alpha}) \qquad (\alpha, \beta, h = 1, \ldots, m).$$

We have now to consider the consistency of these equations and (40). However, both sets of equations may be written

$$\frac{\partial}{\partial s_\alpha}(t_h^\beta) = \sum_{\sigma}^{1\ldots m} t_h^\sigma\, (\gamma_{\beta\sigma\alpha} - \delta_{\alpha\beta}\, B_\sigma + B_\beta\, \delta_{\sigma\alpha}) \qquad \left(\begin{matrix} \alpha = 1, \ldots, n; \\ \beta,\ h = 1, \ldots, m \end{matrix} \right). \tag{47}$$

When we apply conditions of integrability of the form (41) to these equations, we get equations of the form $\sum_{\sigma}^{1\ldots m} t_h^{\sigma} A_{\sigma\beta\alpha e} = 0$, where $A_{\sigma\beta\alpha e}$ is the left-hand member of (48). Since equations of this form hold for $h = 1, \ldots, m$ it follows that we must have

$$
\begin{aligned}
&\gamma_{\beta\sigma,\alpha e} + \sum_{k}^{m+1,\ldots,n} (\gamma_{\beta k e}\,\gamma_{k\sigma\alpha} - \gamma_{\beta k\alpha}\,\gamma_{k\sigma e}) + (\delta_{e\beta}\,\delta_{\sigma\alpha} - \delta_{\alpha\beta}\,\delta_{\sigma e}) \sum_{i}^{1\ldots m} B_i^2 \\
&+ \delta_{\sigma\alpha}\left[\frac{\partial B_\beta}{\partial s_e} + \sum_{i}^{1\ldots m} B_i\,(\gamma_{i\beta e} - \delta_{ie}\,B_\beta)\right] \\
&\qquad - \delta_{\beta\alpha}\left[\frac{\partial B_\sigma}{\partial s_e} + \sum_{i}^{1\ldots m} B_i\,(\gamma_{i\sigma e} - \delta_{ie}\,B_\sigma)\right] \\
&- \delta_{\sigma e}\left[\frac{\partial B_\beta}{\partial s_\alpha} + \sum_{i}^{1\ldots m} B_i\,(\gamma_{i\beta\alpha} - \delta_{i\alpha}\,B_\beta)\right] \\
&\qquad + \delta_{\beta e}\left[\frac{\partial B_\sigma}{\partial s_\alpha} + \sum_{i}^{1\ldots m} B_i\,(\gamma_{i\sigma\alpha} - \delta_{i\alpha}\,B_\sigma)\right] = 0,
\end{aligned}
\tag{48}
$$

for $\beta, \sigma = 1, \ldots, m$; $\alpha, e = 1, \ldots, n$; $\alpha \neq e$.

For β, σ, α, e all different, equation (48) becomes, in consequence of (36) and (38),

$$
\gamma_{\beta\sigma,\alpha e} = 0 \qquad (\beta, \sigma = 1, \ldots, m;\ \alpha, e = 1, \ldots, n;\ \beta, \sigma, \alpha, e \neq). \tag{49}
$$

If we take $\sigma = \alpha$; $\beta, \alpha, e \neq$ or $\beta = \alpha$; $\sigma, \alpha, e \neq$, we obtain from (48) equations of the form

$$
\frac{\partial B_\beta}{\partial s_e} = \sum_{i}^{1\ldots m} B_i\,(\gamma_{\beta ie} + \delta_{ie}\,B_\beta) - \left(\gamma_{\beta\alpha,\alpha e} + \sum_{k}^{m+1\ldots n} \delta_{ke}\,\gamma_{\beta kk}\,\gamma_{k\alpha\alpha}\right) \tag{50}
$$

$$
\left(\begin{matrix}\beta, i, \alpha = 1, \ldots, m;\\ e = 1, \ldots, n;\ \beta \neq e\end{matrix}\right).
$$

If we take $\sigma = \beta$, equation (48) vanishes identically.

If we take $\sigma = \alpha$, $\beta = e$, or $\sigma = e$, $\beta = \alpha$, we get an equation of the form

$$
\begin{aligned}
&\frac{\partial B_e}{\partial s_e} + \frac{\partial B_\alpha}{\partial s_\alpha} - B_e^2 - B_\alpha^2 + \sum_{i}^{1\ldots m} B_i\,[\gamma_{iee} + \gamma_{iaa} + B_i] + \gamma_{ea,ae} \\
&\qquad - \sum_{k}^{m+1\ldots n} \gamma_{kee}\,\gamma_{kaa} = 0 \qquad (\alpha, e = 1, \ldots, m;\ \alpha, e \neq).
\end{aligned}
\tag{51}
$$

From (50) and (37) it follows that we must have

$$(52)\qquad \gamma_{\beta i,\,ie} = \gamma_{\beta\alpha,\,\alpha e} \quad (\alpha, \beta, i = 1, \ldots, m; e = 1, \ldots, n; \alpha, \beta, i, e \neq).$$

If this equation is written in the form $\gamma_{ei,\,i\beta} = \gamma_{ea,\,a\beta}$ it follows that from (20), (36), (37) and (38) this equation is satisfied identically for $e = m+1, \ldots, n$.

In consequence of (37) and (52) we put for the sake of brevity

$$(53)\qquad \begin{aligned} \gamma_{\beta e} &= \gamma_{\beta i,\,ie} + \gamma_{\beta ee}\,\gamma_{eii} \\ \gamma_{\beta\alpha} &= \gamma_{\beta i,\,i\alpha} \end{aligned} \qquad (\beta, i, \alpha = 1, \ldots, m; e = m+1, \ldots, n; \beta, i, \alpha \neq)$$

and write equations (50) in the form

$$(54)\qquad \frac{\partial B_\beta}{\partial s_e} = \sum_i^{1\ldots m} B_i(\gamma_{\beta ie} + \delta_{ie} B_\beta) - \gamma_{\beta e}(\beta, i = 1, \ldots, m;\ e = 1, \ldots, n; \beta, e \neq).$$

When we apply to (54) conditions of integrability of the form (41), we obtain

$$\begin{aligned} &\sum_i^{1\ldots m} B_i \Big[\gamma_{\beta i,\,ae} + \delta_{ie}\,\gamma_{\beta a} - \delta_{ia}\,\gamma_{\beta e} + \gamma_{\beta ae}(\gamma_{i\beta\beta} - \gamma_{iaa}) - \gamma_{\beta ea}(\gamma_{i\beta\beta} - \gamma_{iee}) \\ &\quad + \delta_{ia}\,B_a\,\gamma_{\beta ae} - \delta_{ie}\,B_e\,\gamma_{\beta ea} + \sum_{k\neq\alpha}^{m+1\ldots n} \gamma_{\beta ke}\,\gamma_{kia} - \sum_{k\neq e}^{m+1\ldots n} \gamma_{\beta ka}\,\gamma_{kie}\Big] \\ (55)\quad &+ \sum_i \Big[\frac{\partial B_i}{\partial s_i}(\delta_{ie}\,\gamma_{\beta ia} - \delta_{ia}\,\gamma_{\beta ie}) + B_i\,B_\beta\Big(\sum_j^{1\ldots m}(\delta_{ja}\,\gamma_{jie} - \delta_{je}\,\gamma_{jia}) + \gamma_{iae} - \gamma_{iea}\Big)\Big] \\ &\quad + (\gamma_{\beta ae} - \gamma_{\beta ea})\Big(\frac{\partial B_\beta}{\partial s_\beta} - B_\beta^2\Big) + B_\beta\Big(\sum_{i\neq a}^{1\ldots m} \delta_{ie}\,\gamma_{ia} - \sum_{i\neq e}^{1\ldots m} \delta_{ia}\,\gamma_{ie}\Big) \\ &\quad + \frac{\partial \gamma_{\beta e}}{\partial s_a} - \frac{\partial \gamma_{\beta a}}{\partial s_e} + \sum_{j\neq a}^{1\ldots m} \gamma_{\beta je}\,\gamma_{ja} - \sum_{j\neq e}^{1\ldots m} \gamma_{\beta ja}\,\gamma_{je} + \sum_{k\neq\beta}^{1\ldots n} \gamma_{\beta k}(\gamma_{kea} - \gamma_{kae}) = 0. \end{aligned}$$

For $\alpha, e = m+1, \ldots, n$ we have from (55), in consequence of (36), (38) and (49),

$$(56)\quad \frac{\partial \gamma_{\beta e}}{\partial s_\alpha} - \frac{\partial \gamma_{\beta\alpha}}{\partial s_e} + \sum_j^{1\ldots m} \gamma_{\beta je}\gamma_{j\alpha} - \sum_j^{1\ldots m} \gamma_{\beta j\alpha}\gamma_{je} + \sum_k^{m+1\ldots n} \gamma_{\beta k}(\gamma_{ke\alpha} - \gamma_{k\alpha e}) = 0.$$

For $\alpha = 1, \ldots, m;\ e = m+1, \ldots, n$ we have, in consequence of (36), (37), (38) and (53),

$$\frac{\partial \gamma_{\beta e}}{\partial s_\alpha} - \frac{\partial \gamma_{\beta\alpha}}{\partial s_e} + \sum_{j\neq\alpha}^{1\ldots m} \gamma_{\beta je}\gamma_{j\alpha} - \sum_j^{1\ldots m} \gamma_{\beta j\alpha}\gamma_{je} + \sum_{k\neq\beta}^{1\ldots n} \gamma_{\beta k}(\gamma_{ke\alpha} - \gamma_{k\alpha e})$$

$$+ \gamma_{\beta\alpha e}\Big[\frac{\partial B_\beta}{\partial s_\beta} - B_\beta^2 - \frac{\partial B_\alpha}{\partial s_\alpha} + B_\alpha^2 + \sum_i^{1\ldots m} B_i(\gamma_{i\beta\beta} - \gamma_{i\alpha\alpha})\Big] - B_\beta\gamma_{\alpha e} + B_\alpha B_\beta \gamma_{e\alpha\alpha} = 0.$$

By means of (51) this is reducible to

$$\frac{\partial \gamma_{\beta e}}{\partial s_\alpha} - \frac{\partial \gamma_{\beta\alpha}}{\partial s_e} + \sum_{j\neq\alpha}^{1\ldots m} \gamma_{\beta je}\gamma_{j\alpha} - \sum_j^{1\ldots m} \gamma_{\beta j\alpha}\gamma_{je} + \sum_{k\neq\beta}^{1\ldots n} \gamma_{\beta k}(\gamma_{ke\alpha} - \gamma_{k\alpha e})$$

$$(57)\qquad + \gamma_{\beta\alpha e}(\gamma_{i\alpha,\,\alpha i} - \gamma_{i\beta,\,\beta i}) - B_\beta\gamma_{\alpha e} + B_\beta B_\alpha \gamma_{e\alpha\alpha} = 0.$$

In like manner for $\alpha, e = 1, \ldots, m$, we have

$$\frac{\partial \gamma_{\beta e}}{\partial s_\alpha} - \frac{\partial \gamma_{\beta\alpha}}{\partial s_e} + \sum_{j\neq\alpha}^{1\ldots m} \gamma_{\beta je}\gamma_{j\alpha} - \sum_{j\neq e}^{1\ldots m} \gamma_{\beta j\alpha}\gamma_{je} + \sum_{k\neq\beta}^{1\ldots m} \gamma_{\beta k}(\gamma_{ke\alpha} - \gamma_{k\alpha e})$$

$$(58)\qquad + \gamma_{\beta\alpha e}(\gamma_{e\alpha,\,\alpha e} - \gamma_{e\beta,\,\beta e}) - \gamma_{\beta e\alpha}(\gamma_{\alpha e,\,e\alpha} - \gamma_{\alpha\beta,\,\beta\alpha}) = 0.$$

6. **Reduced form of conditions (56), (57) and (58).** From (18) we have

$$(59)\qquad \gamma_{\beta i,\,ie} = B_{qr,\,st}\lambda_\beta^q \lambda_i^r \lambda_i^s \lambda_e^t.$$

and by differentiation we have for α different from β, i and e

$$\frac{\partial}{\partial s_\alpha}\gamma_{\beta i,ie} = B_{qr,stp}\lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_e^t\lambda_\alpha^p + \sum_j^{1\ldots n}[\gamma_{ji,ie}\gamma_{\beta j\alpha} + \gamma_{\beta j,ie}\gamma_{ij\alpha} + \gamma_{\beta i,je}\gamma_{ij\alpha} + \gamma_{\beta i,ij}\gamma_{ej\alpha}] \quad \left(\begin{matrix}\beta, i = 1\ldots m;\ \alpha, e = 1,\ldots,n;\\ \beta, i, \alpha, e \neq\end{matrix}\right). \tag{60}$$

Bianchi has established the following identity:*

$$B_{qr,stp} + B_{qr,tps} + B_{qr,pst} = 0. \tag{61}$$

Making use of (61) and the well known identities

$$B_{qr,st} = -B_{qr,ts} = -B_{rq,st}, \tag{62}$$

we obtain

$$\frac{\partial}{\partial s_\alpha}\gamma_{\beta i,ie} - \frac{\partial}{\partial s_e}\gamma_{\beta i,i\alpha} = B_{qr,pts}\lambda_\beta^q\lambda_i^r\lambda_\alpha^p\lambda_e^t\lambda_i^s + \sum_j^{1\ldots n}[\gamma_{ji,ie}\gamma_{\beta j\alpha} + (\gamma_{\beta j,ie} + \gamma_{\beta i,je})\gamma_{ij\alpha} - \gamma_{ji,i\alpha}\gamma_{\beta je} - (\gamma_{\beta j,i\alpha} + \gamma_{\beta i,j\alpha})\gamma_{ije} + (\gamma_{j\alpha e} - \gamma_{je\alpha})\gamma_{\beta i,ij}]. \tag{63}$$

Since β, i, α, e are different, we have from (49)

$$\gamma_{\beta i,\alpha e} = B_{qr,pt}\lambda_\beta^q\lambda_i^r\lambda_\alpha^p\lambda_e^t = 0.$$

Differentiate with respect to x^s, multiply by λ_i^s and sum for s; we get

$$B_{qr,pts}\lambda_\beta^q\lambda_i^r\lambda_\alpha^p\lambda_e^t\lambda_i^s + \sum_j^{1\ldots n}[\gamma_{ji,\alpha e}\gamma_{\beta ji} + \gamma_{\beta j,\alpha e}\gamma_{iji} + \gamma_{\beta i,je}\gamma_{\alpha ji} + \gamma_{\beta i,\alpha j}\gamma_{eji}] = 0.$$

* *Lezioni*, vol. 1, p. 351.

Hence in consequence of (49) we obtain

$$\frac{\partial}{\partial s_\alpha}\gamma_{\beta i,ie}-\frac{\partial}{\partial s_e}\gamma_{\beta i,i\alpha}=\gamma_{\beta\alpha i}(\gamma_{i\alpha,ae}-\gamma_{i\beta,\beta e})+\gamma_{\beta ei}(\gamma_{i\beta,\beta\alpha}-\gamma_{ie,e\alpha})$$

$$+\gamma_{\alpha ii}(\gamma_{\beta\alpha,ae}-\gamma_{\beta i,ie})+\gamma_{eii}(\gamma_{\beta i,i\alpha}-\gamma_{\beta e,e\alpha})+\gamma_{eia}\gamma_{\beta e,ei}$$

$$(64)\qquad +\sum_{j\neq\alpha}^{1\ldots m}\gamma_{ji,ie}\gamma_{\beta ja}+\gamma_{i\beta,\beta e}\gamma_{\beta ia}+\gamma_{\alpha i,ie}(\gamma_{\beta\alpha a}-\gamma_{\beta ee})-\gamma_{\alpha ie}\gamma_{\beta\alpha,\alpha i}$$

$$-\sum_{j\neq e}^{1\ldots m}\gamma_{ji,i\alpha}\gamma_{\beta je}-\gamma_{i\beta,\beta\alpha}\gamma_{\beta ie}+\sum_{j}^{1\ldots n}(\gamma_{jae}-\gamma_{jea})\gamma_{\beta i,ij}$$

$$+\sum_{k\neq\alpha,e}^{m+1\ldots n}\gamma_{\beta k,ae}\gamma_{kii}+\gamma_{\beta\alpha,ie}\gamma_{iaa}-\gamma_{\beta e,ia}\gamma_{iee}.$$

For $\alpha, e=m+1,\ldots,n, \alpha\neq e$, and $\beta, i=1,\ldots,m, \beta\neq i$, we have from (20), in consequence of (36), (37) and (38),

$$\frac{\partial}{\partial s_\alpha}\gamma_{\beta ee}=\gamma_{\beta e,ea}+\sum_{j}^{1\ldots m}\gamma_{\beta ja}\gamma_{jee}+\gamma_{aee}(\gamma_{\beta\alpha a}-\gamma_{\beta ee}),$$

$$(65)\qquad \frac{\partial}{\partial s_\alpha}\gamma_{eii}=\gamma_{ei,ia}-\gamma_{eii}\gamma_{aii}+\gamma_{aei}(\gamma_{iee}-\gamma_{iaa})-\sum_{k}^{m+1\ldots n}\gamma_{kea}\gamma_{kii},$$

$$\gamma_{\beta\alpha,ie}=\gamma_{aei}(\gamma_{\beta ee}-\gamma_{\beta aa}),$$

$$\gamma_{\beta k,ae}=\gamma_{kae}(\gamma_{\beta kk}-\gamma_{\beta aa})-\gamma_{kea}(\gamma_{\beta kk}-\gamma_{\beta ee})$$

$$(k=m+1,\ldots,n;\ \alpha,e,k\neq).$$

By means of these results, equations (64) and (53), defining the function $\gamma_{\beta e}$, we find that (56) is reducible to

$$(66)\qquad \gamma_{eii}\gamma_{\beta\alpha}-\gamma_{\alpha ii}\gamma_{\beta e}=0 \qquad (\beta, i=1,\ldots,m;\ \alpha, e=m+1,\ldots,n).$$

[1] For $\alpha, \beta, i=1,\ldots,m; e=m+1,\ldots,n$ we have from (20) in consequence of (36), (37) and (38).

$$\frac{\partial \gamma_{\beta ee}}{\partial s_\alpha} = \gamma_{\beta e, e\alpha} - \gamma_{\beta ee}\gamma_{\alpha ee} - \sum_j^{1\ldots m} \gamma_{j\beta\alpha}\gamma_{jee},$$

$$\frac{\partial \gamma_{eii}}{\partial s_\alpha} = \gamma_{ei, i\alpha} - \sum_k^{m+1\ldots n} \gamma_{ke\alpha}\gamma_{kii}, \tag{67}$$

$$\gamma_{\beta k, \alpha e} = \gamma_{ke\alpha}(\gamma_{\beta ee} - \gamma_{\beta kk}) \qquad (k = m+1, \ldots, n;\ k \neq e).$$

By means of these results, equations (64) and (53), we find that (57) is reducible to

$$\gamma_{eii}\gamma_{\beta\alpha} - B_\beta \gamma_{\alpha e} + B_\beta B_\alpha \gamma_{eii} = 0 \qquad \begin{pmatrix}\alpha, \beta, i = 1, \ldots, m;\\ e = m+1, \ldots, n\end{pmatrix}. \tag{68}$$

For $\alpha, \beta, e = 1, \ldots, m;\ \alpha, \beta, e \neq$ we have

$$\frac{\partial \gamma_{\beta e}}{\partial s_\alpha} = \frac{\partial \gamma_{\beta\alpha, \alpha e}}{\partial s_\alpha} = B_{qr,stp}\,\lambda_\beta^q \lambda_\alpha^r \lambda_\alpha^s \lambda_e^t \lambda_\alpha^p + \sum_{j\neq e}^{1\ldots m} \gamma_{je}\gamma_{\beta j\alpha} + \sum_{j\neq\beta}^{1\ldots m} \gamma_{\beta j}\gamma_{ej\alpha} + \gamma_{e\alpha,\alpha e}\gamma_{\beta e\alpha} + \gamma_{\beta\alpha,\alpha\beta}\gamma_{e\beta\alpha}. \tag{69}$$

Substituting in (58), we get

$$B_{qr,stp}(\lambda_\beta^q \lambda_\alpha^r \lambda_\alpha^s \lambda_e^t \lambda_\alpha^p - \lambda_\beta^q \lambda_e^r \lambda_e^s \lambda_\alpha^t \lambda_e^p) = 0 \qquad \begin{pmatrix}\beta, \alpha, e = 1, \ldots, m,\\ \beta, \alpha, e \neq\end{pmatrix}. \tag{70}$$

7. **Consistency of equations (50) and (51).** If we take the three equations of the form (51) involving three distinct indices α, β and i, we obtain

$$\frac{\partial B_\beta}{\partial s_\beta} = B_\beta^2 - \sum_j^{1\ldots m}(B_j\gamma_{j\beta\beta} + \tfrac{1}{2}B_j^2) + \tfrac{1}{2}(\gamma_{i\alpha,\alpha i} - \gamma_{i\beta,\beta i} - \gamma_{\alpha\beta,\beta\alpha}) + \frac{1}{2}\sum_k^{m+1\ldots n}\gamma_{kii}^2. \tag{71}$$

Expressing the consistency of this equation and (50) for $e = m+1, \ldots, n$ by means of equations of the form (41), we reduce the resulting equations to the form

$$\frac{1}{2}\frac{\partial}{\partial s_e}(\gamma_{i\alpha,\alpha i}-\gamma_{i\beta,\beta i}-\gamma_{\alpha\beta,\beta\alpha})+\frac{\partial\gamma_{\beta e}}{\partial s_\beta}+\sum_k^{m+1\ldots n}\gamma_{kii}\frac{\partial\gamma_{kii}}{\partial s_e}+\sum_j^{1\ldots m}\gamma_{j\beta\beta}\gamma_{je}$$

$$+\gamma_{\beta ee}\gamma_{\beta e}+2\sum_j^{1\ldots m}\gamma_{\beta je}\gamma_{j\beta}+\sum_k^{m+1\ldots n}\gamma_{ke\beta}\gamma_{k\beta}-B_\beta\gamma_{\beta e} \tag{72}$$

$$+\gamma_{eii}\Big[B_\beta^2-\frac{1}{2}\sum_j^{1\ldots m}B_j^2+\tfrac{1}{2}(\gamma_{i\alpha,\alpha i}-\gamma_{i\beta,\beta i}-\gamma_{\alpha\beta,\beta\alpha})+\frac{1}{2}\sum_k^{m+1\ldots n}\gamma_{kii}^2\Big]=0.$$

From the equation

$$\gamma_{i\alpha,\alpha i}=B_{qr,st}\lambda_i^q\lambda_\alpha^r\lambda_\alpha^s\lambda_i^t,$$

we find

$$\frac{\partial\gamma_{i\alpha,\alpha i}}{\partial s_e}=B_{qr,stp}\lambda_i^q\lambda_\alpha^r\lambda_\alpha^s\lambda_i^t\lambda_e^p+2\sum_j^{1\ldots m}(\gamma_{j\alpha}\gamma_{\alpha je}+\gamma_{ji}\gamma_{ije}) \tag{73}$$

$$+2\gamma_{\alpha i,ie}\gamma_{\alpha ee}+2\gamma_{i\alpha,\alpha e}\gamma_{iee}.$$

In like manner we have

$$\frac{\partial\gamma_{\beta i,ie}}{\partial s_\beta}=B_{qr,stp}\lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_e^t\lambda_\beta^p-\sum_j^{1\ldots n}\gamma_{ji,ie}\gamma_{j\beta\beta}-\gamma_{i\beta,\beta e}\gamma_{i\beta\beta} \tag{74}$$

$$+\gamma_{\beta i,i\beta}\gamma_{e\beta\beta}+\sum_k^{m+1\ldots n}\gamma_{\beta i,ik}\gamma_{ek\beta}.$$

Hence making use of (20), we find

$$\frac{\partial\gamma_{\beta e}}{\partial s_\beta}=\frac{\partial}{\partial s_\beta}(\gamma_{\beta i,ie}+\gamma_{\beta ee}\gamma_{eii})=B_{qr,stp}\lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_e^t\lambda_\beta^p$$

$$-\sum_j^{1\ldots m}\gamma_{je}\gamma_{j\beta\beta}+\sum_k^{m+1\ldots n}(\gamma_{\beta k}\gamma_{ek\beta}-\gamma_{ki,ie}\gamma_{k\beta\beta})+\gamma_{\beta i,i\beta}\gamma_{e\beta\beta} \tag{75}$$

$$+\gamma_{\beta ee}\gamma_{\beta i,ie}+\gamma_{eii}(\gamma_{\beta e,e\beta}-\frac{\partial\gamma_{eii}}{\partial s_e}-\gamma_{eii}^2-\gamma_{\beta ee}^2-\sum_k^{m+1\ldots n}\gamma_{kee}\gamma_{k\beta\beta}).$$

Substituting in (72), we obtain

$$
\begin{aligned}
&-B_\beta\,\gamma_{\beta e}+\sum_k^{m+1\ldots n}\gamma_{kii}\,[-\gamma_{kii}\,\gamma_{eii}+\gamma_{eki}\,(\gamma_{ikk}-\gamma_{iee})]\\
&+\gamma_{eii}\,(\gamma_{\beta e,\,e\beta}-\gamma_{ei,\,ie})+\gamma_{\beta i,\,i\beta}\,\gamma_{eii}+\gamma_{eii}\Big[B_\beta^2-\frac{1}{2}\sum_j^{1\ldots m}B_j^2+\frac{1}{2}\sum_k^{m+1\ldots n}\gamma_{kii}^2\\
&+\tfrac{1}{2}(\gamma_{i\alpha,\,\alpha i}-\gamma_{i\beta,\,\beta i}-\gamma_{\alpha\beta,\,\beta\alpha})\Big]+B_{qr,\,stp}[\lambda_\beta^q\,\lambda_i^r\,\lambda_i^s\,\lambda_e^t\,\lambda_\beta^p\\
&+\tfrac{1}{2}(\lambda_i^q\,\lambda_\alpha^r\,\lambda_\alpha^s\,\lambda_i^t-\lambda_i^q\,\lambda_\beta^r\,\lambda_\beta^s\,\lambda_i^t-\lambda_\alpha^q\,\lambda_\beta^r\,\lambda_\beta^s\,\lambda_\alpha^t)\,\lambda_e^p]=0.
\end{aligned}
\tag{76}
$$

In consequence of (61) and (62) we have

$$
B_{qr,\,stp}\,\lambda_i^q\,\lambda_\alpha^r\,\lambda_\alpha^s\,\lambda_i^t\,\lambda_e^p=B_{qr,\,stp}\,(\lambda_\alpha^q\,\lambda_i^r\,\lambda_i^s\,\lambda_e^t\,\lambda_\alpha^p+\lambda_i^q\,\lambda_\alpha^r\,\lambda_\alpha^s\,\lambda_e^t\,\lambda_i^p).
\tag{77}
$$

From (52) and (74) it follows that

$$
\begin{aligned}
&B_{qr,\,stp}\,(\lambda_\beta^q\,\lambda_i^r\,\lambda_i^s\,\lambda_e^t\,\lambda_\beta^p-\lambda_\beta^q\,\lambda_\alpha^r\,\lambda_\alpha^s\,\lambda_e^t\,\lambda_\beta^p)\\
&\qquad=\sum_k^{m+1\ldots n}(\gamma_{ki,\,ie}-\gamma_{k\alpha,\,\alpha e})\,\gamma_{k\beta\beta}+\gamma_{eii}\,(\gamma_{\beta\alpha,\,\alpha\beta}-\gamma_{\beta i,\,i\beta}).
\end{aligned}
\tag{78}
$$

From (20) it follows that

$$
\gamma_{ki,\,ie}-\gamma_{k\alpha,\,\alpha e}=\gamma_{eki}\,(\gamma_{iee}-\gamma_{ikk})-\gamma_{ek\alpha}\,(\gamma_{\alpha ee}-\gamma_{\alpha kk})\qquad(e\neq k).
\tag{79}
$$

Substituting in (76) from (77), (78), (79) and similar expressions we have

$$
\begin{aligned}
&\gamma_{eii}\Big[B_\beta^2-\frac{1}{2}\sum_j^{1\ldots m}B_j^2-\tfrac{1}{2}\,(\gamma_{i\alpha,\,\alpha i}-\gamma_{i\beta,\,\beta i}-\gamma_{\alpha\beta,\,\beta\alpha})\\
&\qquad-\frac{1}{2}\sum_k^{m+1\ldots n}\gamma_{kii}^2\Big]-B_\beta\,\gamma_{\beta e}=0.
\end{aligned}
\tag{80}
$$

In arriving at this result we made use of the fact that $\sum_{k}^{m+1\ldots n} \gamma_{kii}\gamma_{ek\beta}(\gamma_{\beta ee}-\gamma_{\beta kk})=0$. For if e and k belong to different roots of (2), then $\gamma_{ek\beta}=0$ and if to the same root the expression in parenthesis vanishes, in consequence of equations analogous to (37). Hence (76) reduces to (80).

When we express the consistency of (71) and

$$\frac{\partial B_\beta}{\partial s_\alpha} = \sum_{j}^{1\ldots m} B_j\,\gamma_{\beta j\alpha} + B_\alpha B_\beta - \gamma_{\alpha\beta} \qquad (\alpha = 1, \ldots, m;\ \alpha, \beta \neq), \tag{81}$$

we obtain

$$\frac{1}{2}\frac{\partial}{\partial s_\alpha}(\gamma_{i\alpha,\alpha i}-\gamma_{i\beta,\beta i}-\gamma_{\alpha\beta,\beta\alpha}) + \sum_{k}^{m+1\ldots n}\gamma_{kii}\frac{\partial\gamma_{kii}}{\partial s_\alpha} + \frac{\partial\gamma_{\alpha\beta}}{\partial s_\beta}$$

$$+\gamma_{\alpha\beta\beta}(\gamma_{i\alpha,\alpha i}-\gamma_{i\beta,\beta i}) + \sum_{j\neq\alpha}^{1\ldots m}\gamma_{j\beta\beta}\gamma_{j\alpha} + \sum_{j\neq\beta}^{1\ldots m}(\gamma_{j\alpha\beta}-2\gamma_{j\beta\alpha})\gamma_{\beta j} = 0.$$

Proceeding as in the case of (72), we obtain

$$B_{qr,stp}(\lambda_i^q\lambda_\alpha^r\lambda_\alpha^s\lambda_i^t\lambda_\alpha^p - \lambda_i^q\lambda_\beta^r\lambda_\beta^s\lambda_i^t\lambda_\alpha^p - \lambda_\beta^q\lambda_\alpha^r\lambda_\alpha^s\lambda_\beta^t\lambda_\alpha^p + 2\lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_\alpha^t\lambda_\beta^p) = 0.$$

In consequence of (61) and (62) we have

$$B_{qr,stp}\lambda_i^q\lambda_\beta^r\lambda_\beta^s\lambda_i^t\lambda_\alpha^p = B_{qr,stp}(\lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_\alpha^t\lambda_\beta^p + \lambda_i^q\lambda_\beta^r\lambda_\beta^s\lambda_\alpha^t\lambda_i^p).$$

Hence the preceding equation becomes

$$B_{qr,stp}[\lambda_i^q\lambda_\alpha^r\lambda_\alpha^s\lambda_i^t\lambda_\alpha^p - \lambda_i^q\lambda_\beta^r\lambda_\beta^s\lambda_\alpha^t\lambda_i^p - \lambda_\beta^q\lambda_\alpha^r\lambda_\alpha^s\lambda_\beta^t\lambda_\alpha^p + \lambda_\beta^q\lambda_i^r\lambda_i^s\lambda_\alpha^t\lambda_\beta^p] = 0. \tag{82}$$

8. **Particular solutions of the problem.** If the functions $\gamma_{eii}=0$, for $i=1,\ldots,m$; $e=m+1,\ldots,n$, that is, if

$$\alpha_{rst}\lambda_e^r\lambda_i^s\lambda_i^t = 0 \qquad (i=1,\ldots,m;\ e=m+1,\ldots,n), \tag{83}$$

it follows from (67) that

$$\gamma_{\alpha i,ie} = 0 \qquad (\alpha, i = 1, \ldots, m;\ e = m+1, \ldots, n), \tag{84}$$

and from (53) that $\gamma_{ae} = 0$. Hence the conditions (66), (68) and (80) are satisfied identically. The conditions (49) may be written

$$B_{qr,st}\,\lambda_\beta^q\,\lambda_i^r\,\lambda^s\,\lambda_e^t = 0 \quad (\beta, i = 1, \ldots, m;\ \alpha, e = 1, \ldots, n;\ \beta, i, \alpha, e \neq). \tag{85}$$

In addition to (84) equations (52) for $m > 3$ lead to conditions of the form

$$B_{qr,st}\,(\lambda_\beta^q\,\lambda_\alpha^r\,\lambda_\alpha^s\,\lambda_i^t - \lambda_\beta^q\,\lambda_j^r\,\lambda_j^s\,\lambda_i^t) = 0 \quad (\beta, \alpha, i, j = 1, \ldots, m;\ \beta, \alpha, i, j \neq). \tag{86}$$

In addition we have (70) and (82). Conversely, if equations (83) are satisfied, we have $\gamma_{eii} = 0$ for $i = 1, \ldots, m$; $e = m+1, \ldots, n$. Hence:

When the solutions of (3) *satisfy equations* (29), (39), *and in addition sets of equations of the forms* (70), (82), (83), (85) *and* (86) *for each multiple root of the third or higher order of equations* (2), *there exists a normal n-uple in the space.*

Equations (50) and (51) are satisfied by $B_\alpha = 0$ ($\alpha = 1, \ldots, m$), if

$$\gamma_{\beta e} = 0 \qquad (\beta = 1, \ldots, m;\ e = 1, \ldots, n;\ \beta, e \neq),$$

$$\gamma_{a\beta,\beta\alpha} - \sum_k^{m+1 \ldots n} \gamma_{kii}^2 = 0. \tag{87}$$

Conversely, it follows from (68) that if equations (87) are satisfied and $\gamma_{eii} \neq 0$ for any $e = m+1, \ldots, n$, then the only solutions of (50) and (51) are $B_\alpha = 0$.

From (24) and (27) we have

$$\alpha_{rs|t}\,\lambda_\beta^r\,\lambda_e^s\,\lambda_e^t + \alpha_{rs}(\lambda_e^r\,\lambda_e^s - \lambda_\beta^r\,\lambda_\beta^s)\,\gamma_{\beta ee} = 0,$$

$$\alpha_{rs|t}\,\lambda_e^r\,\lambda_i^s\,\lambda_i^t + \alpha_{rs}(\lambda_i^r\,\lambda_i^s - \lambda_e^r\,\lambda_e^s)\,\gamma_{eii} = 0.$$

By means of equations of this form and (53) equations (87) are expressible in terms of α_{rs}, its first covariant derivative, $B_{qr,st}$ and λ's. Hence:

When the solutions of (3) *satisfy* (29), (39), (49) *and equations of the form* (87) *for each multiple root of* (2) *of the third or higher order, there exists a normal n-uple in the space.*

If $\gamma_{ae} = 0$, $\gamma_{eii} \neq 0$, for a particular $\alpha = 1, \ldots, m$ and a particular $e = m+1, \ldots, n$, from (66) and (68), on the assumption that all of the B's are different from zero, it follows that we must have

$$\gamma_{ae} = 0 \qquad (\alpha = 1, \ldots, m;\ e = m+1, \ldots, n), \tag{88}$$

$$B_\alpha B_\beta + \gamma_{\beta\alpha} = 0 \qquad (\alpha, \beta = 1, \ldots, m;\ \alpha, \beta \neq). \tag{89}$$

Operating on (89) with $\frac{\partial}{\partial s_e}$ for $e = m+1, \ldots, n$ and making use of (54), we obtain

$$\gamma_{\beta ae}(B_\alpha^2 - B_\beta^2) - \sum_{j \neq \alpha}^{1 \ldots m} \gamma_{\alpha j}\, \gamma_{\beta je} - \sum_{j \neq \beta}^{1 \ldots m} \gamma_{\beta j}\, \gamma_{\alpha je} + \frac{\partial \gamma_{\alpha\beta}}{\partial s_e} = 0.$$

From (57) we have also

$$-\frac{\partial \gamma_{\alpha\beta}}{\partial s_e} + \sum_{j \neq \alpha}^{1 \ldots m} \gamma_{\alpha j}\, \gamma_{\beta je} + \sum_{j \neq \beta}^{1 \ldots m} \gamma_{\beta j}\, \gamma_{\alpha je} + \gamma_{\beta ae}(\gamma_{ia,\alpha i} - \gamma_{i\beta,\beta i}) - 2\gamma_{eii}\, \gamma_{\beta\alpha} = 0.$$

From these equations it follows that

$$\gamma_{\beta ae}(B_\alpha^2 - B_\beta^2 + \gamma_{ia,\,\alpha i} - \gamma_{i\beta,\beta i}) - 2\gamma_{eii}\, \gamma_{\beta\alpha} = 0. \tag{90}$$

Subtracting from (80) the equation obtained by interchanging α and β in this equation, we obtain

$$\gamma_{eii}(B_\alpha^2 - B_\beta^2 + \gamma_{ia,\alpha i} - \gamma_{i\beta,\beta i}) = 0.$$

From this equation and (90), it follows that $\gamma_{\beta\alpha} = 0$, which is inconsistent with (89).

9. **General solution.** We consider finally the general case when (88) is not satisfied. If in conformity with (66) we put

$$\gamma_{\beta e} = \gamma_{eii}\, \sigma_\beta \qquad (\beta, i = 1, \ldots, m;\ e = m+1, \ldots, n), \tag{91}$$

equation (68) becomes

$$\gamma_{\beta\alpha} - B_\beta \sigma_\alpha + B_\beta B_\alpha = 0. \tag{92}$$

Interchange α and β and subtract; then

$$B_\alpha = B_\beta \frac{\sigma_\alpha}{\sigma_\beta}, \tag{93}$$

and (92) may be replaced by

$$B_\beta^2 - \sigma_\beta B_\beta + \frac{\sigma_\beta}{\sigma_\alpha}\gamma_{\beta\alpha} = 0. \tag{94}$$

From equations of this form it follows that $\gamma_{\beta\alpha}/\sigma_\alpha = \gamma_{\beta i}/\sigma_i$ and consequently

$$\sigma_i \gamma_{\beta\alpha} = \sigma_\alpha \gamma_{\beta i} = \sigma_\beta \gamma_{\alpha i}. \tag{95}$$

Moreover, from (91) and (95) we obtain

$$\gamma_{\beta\alpha}\gamma_{ie} = \gamma_{\beta i}\gamma_{\alpha e} = \gamma_{\alpha i}\gamma_{\beta e} \quad (\alpha, \beta, i = 1, \ldots, m; e = m+1, \ldots, n). \tag{96}$$

Equation (80) is reducible by means of (91) and (94) to

$$\sum_j^{1\ldots m} B_j^2 + (\gamma_{i\alpha,\alpha i} - \gamma_{i\beta,\beta i} - \gamma_{\alpha\beta,\beta\alpha}) + \sum_k^{m+1\ldots n} \gamma_{kii}^2 + \frac{2\gamma_{\beta e}}{\gamma_{\alpha e}}\gamma_{\alpha\beta} = 0. \tag{97}$$

Interchanging α and β in these equations and subtracting, we obtain

$$\gamma_{i\alpha,\alpha i} - \gamma_{i\beta,\beta i} + \left(\frac{\gamma_{\beta e}}{\gamma_{\alpha e}} - \frac{\gamma_{\alpha e}}{\gamma_{\beta e}}\right)\gamma_{\alpha\beta} = 0 \tag{98}$$

$$(\alpha, \beta, i = 1, \ldots, m; e = m+1, \ldots, n).$$

Hence (97) reduces to

$$\sum_j^{1\ldots m} B_j^2 - \gamma_{\alpha\beta,\beta\alpha} + \left(\frac{\gamma_{\beta e}}{\gamma_{\alpha e}} + \frac{\gamma_{\alpha e}}{\gamma_{\beta e}}\right)\gamma_{\alpha\beta} + \sum_k^{m+1\ldots n} \gamma_{kii}^2 = 0.$$

By means of equations of the form (94) this is reducible to

$$\sum_{j}^{1\ldots m} \gamma_{je} B_j = \gamma_{eii}\Big[\gamma_{\alpha\beta,\beta\alpha} - \sum_{k}^{m+1\ldots n} \gamma_{kii}^2 + \sum_{j\neq\alpha,\beta}^{1\ldots m} \frac{\gamma_{je}}{\gamma_{ae}}\gamma_{j\alpha}\Big],$$

and this in turn is reducible, by means of equations of the form (93), to

$$B_\alpha \sum_{j}^{1\ldots m} \gamma_{je}^2 = \gamma_{eii}\Big[\Big(\gamma_{\alpha\beta,\beta\alpha} - \sum_{k}^{m+1\ldots n} \gamma_{kii}^2\Big)\gamma_{ae} + \sum_{j\neq\alpha,\beta}^{1\ldots m} \gamma_{je}\gamma_{j\alpha}\Big]. \tag{99}$$

As a consequence we must have

$$(\gamma_{\alpha\beta,\beta\alpha} - \gamma_{\alpha i,i\alpha})\gamma_{ae} = \sum_{j\neq\alpha,i}^{1\ldots m} \gamma_{je}\gamma_{j\alpha} - \sum_{j\neq\alpha,\beta}^{1\ldots m} \gamma_{je}\gamma_{j\alpha} = \gamma_{\beta e}\gamma_{\beta\alpha} - \gamma_{ie}\gamma_{i\alpha},$$

which because of (96) is equivalent to an equation of the form (98).

When the expressions for $B_\alpha (\alpha = 1, \ldots, m)$ from (99) are substituted in (54), (71) and (94), we obtain the set of conditions to be satisfied in addition to those previously found, and thus the problem is reduced to an algebraic one. Recapitulating we have:

If the solutions of (3) *satisfy* (29), *and for each multiple root equations of the form* (39), (66), (70), (82), (85), (86), (96), (98) *and the equations resulting from the substitution of the functions* B_α *given by* (99) *in* (54), (71) *and* (94), *the space admits a normal n-uple.*

From the manner in which these conditions were derived it is clear that while all of them are necessary, they are not necessarily independent. However, they are necessary and sufficient to determine whether the space possesses a normal n-uple.

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Note added in proof. In considering the particular solutions in § 8, I omitted the possibility of one or more of the B's being zero, but not all of them. For $B_i = 0$, we have from (68) $\gamma_{eii}\gamma_{\beta 1} - B_\beta \gamma_{1e} = 0$ $(\beta = 1, \ldots, m;\ e = m+1, \ldots, n)$ and from (54) $\sum_{i}^{1\ldots m} B_i \gamma_{1ie} - \gamma_{1e} = 0$ $(e = 1, \ldots, n)$. When the expression for B_β from the first is substituted in the second, we obtain one set of conditions, and when substituted in (51) and (54) the others to be added to (29), (39), (66), (70), (85) and (86).

RULED SURFACES WITH GENERATORS IN ONE-TO-ONE CORRESPONDENCE*

BY

ERNEST P. LANE

1. INTRODUCTION

The configuration composed of two ruled surfaces whose generators are in one-to-one correspondence occurs frequently in geometry. If we assume that corresponding generators are not coplanar, we find that we can base a projective theory of this configuration on a system of four ordinary linear first order differential equations in four dependent variables. If we choose the fundamental curves of reference on the two surfaces suitably, we are able to reduce this system of equations to a relatively simple canonical form.

The curves which are fundamental for our canonical form we have called intersector curves. They are in some respects similar to the curved asymptotics on a ruled surface. We also define other curves which are analogous to flecnode curves and have some of their properties.

As an application of our theory, we have employed it to investigate Green-reciprocal ruled surfaces. The method proves to be a fruitful one. We are able to generalize some well known theorems concerning ruled surfaces of congruences I and I' which are reciprocal in the sense of G. M. Green. And we discover a geometrical characterization of the directrix congruences of a surface in terms of simple concepts.

2. THE DIFFERENTIAL EQUATIONS

Let the four homogeneous coördinates $y^{(1)}, \ldots, y^{(4)}$ of an arbitrary point P_y on a curve C_y be given as analytic functions of a single independent variable x. And in like manner let the four coördinates of each of three other points P_z, P_ρ, P_σ be given as functions of the same variable x. Let us join P_y and P_z by a straight line l_{yz}; let us also join P_ρ and P_σ by a line $l_{\rho\sigma}$. Then, as x varies, the locus of l_{yz} is a ruled surface R_{yz}, and the locus of $l_{\rho\sigma}$ is a ruled surface $R_{\rho\sigma}$. It is these ruled surfaces that we wish to study.

We shall suppose that P_y, P_z, P_ρ, P_σ are not coplanar, so that corresponding generators l_{yz} and $l_{\rho\sigma}$ do not intersect. Then the determinant

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$$\begin{vmatrix} y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \varrho^{(1)} & \varrho^{(2)} & \varrho^{(3)} & \varrho^{(4)} \\ \sigma^{(1)} & \sigma^{(2)} & \sigma^{(3)} & \sigma^{(4)} \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$\begin{aligned} y' &= c_{11}\, y + c_{12}\, z + a_{11}\, \varrho + a_{12}\, \sigma, \\ z' &= c_{21}\, y + c_{22}\, z + a_{21}\, \varrho + a_{22}\, \sigma, \\ \varrho' &= b_{11}\, y + b_{12}\, z + d_{11}\, \varrho + d_{12}\, \sigma, \\ \sigma' &= b_{21}\, y + b_{22}\, z + d_{21}\, \varrho + d_{22}\, \sigma \end{aligned} \tag{1}$$

so that $(y^{(i)}, z^{(i)}, \varrho^{(i)}, \sigma^{(i)})$ $(i = 1, 2, 3, 4)$ will be four sets of solutions. For example, we may substitute each of these four sets in turn in the first equation of system (1) and then solve the resulting four equations for the coefficients of the first of equations (1). Similarly we may determine the coefficients of each of the other three equations.

System (1) is not uniquely determined when R_{yz} and $R_{\rho\sigma}$ are given. If we introduce new reference curves $C_{\bar{y}}$ and $C_{\bar{z}}$ on R_{yz} by the transformation

$$\begin{aligned} y &= a\bar{y} + b\bar{z}, \\ z &= c\bar{y} + d\bar{z}, \qquad D \equiv ad - bc \neq 0, \end{aligned}$$

and new reference curves $C_{\bar{\rho}}$ and $C_{\bar{\sigma}}$ on $R_{\rho\sigma}$ by the transformation

$$\begin{aligned} \varrho &= \alpha\bar{\varrho} + \beta\bar{\sigma}, \\ \sigma &= \gamma\bar{\varrho} + \delta\bar{\sigma}, \qquad \Delta \equiv \alpha\delta - \beta\gamma \neq 0, \end{aligned}$$

we do not change the surfaces R_{yz} and $R_{\rho\sigma}$; but system (1) goes over into another system of the same form whose coefficients, indicated by dashes, are given by the following equations:

$$
\begin{aligned}
D\bar{c}_{11} &= d(-a' + c_{11}a + c_{12}c) + b(c' - c_{21}a - c_{22}c),\\
D\bar{c}_{12} &= d(-b' + c_{11}b + c_{12}d) + b(d' - c_{21}b - c_{22}d),\\
D\bar{a}_{11} &= d(a_{11}\alpha + a_{12}\gamma) - b(a_{21}\alpha + a_{22}\gamma),\\
D\bar{a}_{12} &= d(a_{11}\beta + a_{12}\delta) - b(a_{21}\beta + a_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
D\bar{c}_{21} &= -c(-a' + c_{11}a + c_{12}c) - a(c' - c_{21}a - c_{22}c),\\
D\bar{c}_{22} &= -c(-b' + c_{11}b + c_{12}d) - a(d' - c_{21}b - c_{22}d),\\
D\bar{a}_{21} &= -c(a_{11}\alpha + a_{12}\gamma) + a(a_{21}\alpha + a_{22}\gamma),\\
D\bar{a}_{22} &= -c(a_{11}\beta + a_{12}\delta) + a(a_{21}\beta + a_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{b}_{11} &= \delta(b_{11}a + b_{12}c) - \beta(b_{21}a + b_{22}c),\\
\Delta\bar{b}_{12} &= \delta(b_{11}b + b_{12}d) - \beta(b_{21}b + b_{22}d),\\
\Delta\bar{d}_{11} &= \delta(-\alpha' + d_{11}\alpha + d_{12}\gamma) + \beta(\gamma' - d_{21}\alpha - d_{22}\gamma),\\
\Delta\bar{d}_{12} &= \delta(-\beta' + d_{11}\beta + d_{12}\delta) + \beta(\delta' - d_{21}\beta - d_{22}\delta),
\end{aligned}
$$

$$
\begin{aligned}
\Delta\bar{b}_{21} &= -\gamma(b_{11}a + b_{12}c) + \alpha(b_{21}a + b_{22}c),\\
\Delta\bar{b}_{22} &= -\gamma(b_{11}b + b_{12}d) + \alpha(b_{21}b + b_{22}d),\\
\Delta\bar{d}_{21} &= -\gamma(-\alpha' + d_{11}\alpha + d_{12}\gamma) - \alpha(\gamma' - d_{21}\alpha - d_{22}\gamma),\\
\Delta\bar{d}_{22} &= -\gamma(-\beta' + d_{11}\beta + d_{12}\delta) - \alpha(\delta' - d_{21}\beta - d_{22}\delta).
\end{aligned}
$$

By a suitable choice of reference curves, the geometrical characterization of which will be furnished later, we are able to reduce system (1) to a simple canonical form. To this end we observe that if we choose a and c as a pair of solutions of the simultaneous differential equations

$$a' = c_{11}a + c_{12}c, \qquad c' = c_{21}a + c_{22}c,$$

we shall then have $\bar{c}_{11} = \bar{c}_{21} = 0$. Likewise we can make $\bar{c}_{12} = \bar{c}_{22} = 0$ by choosing b and d as solutions of

$$b' = c_{11}b + c_{12}d, \qquad d' = c_{21}b + c_{22}d.$$

We can make $\bar{d}_{11} = \bar{d}_{21}$ by choosing α and γ as solutions of

$$\alpha' = d_{11}\,\alpha + d_{12}\,\gamma, \qquad \gamma' = d_{21}\,\alpha + d_{22}\,\gamma,$$

and we can make $\bar{d}_{12} = \bar{d}_{22} = 0$ by choosing β and δ as solutions of

$$\beta' = d_{11}\,\beta + d_{12}\,\delta, \qquad \delta' = d_{21}\,\beta + d_{22}\,\delta.$$

When these reductions have been made, system (1) takes the canonical form

$$\begin{aligned} y' &= a_{11}\,\varrho + a_{12}\,\sigma, & \varrho' &= b_{11}\,y + b_{12}\,z, \\ z' &= a_{21}\,\varrho + a_{22}\,\sigma, & \sigma' &= b_{21}\,y + b_{22}\,z. \end{aligned} \tag{2}$$

Wilczynski has developed the theory of a single ruled surface, for which his fundamental system of equations has the form*

$$\begin{aligned} y'' + p_{11}\,y' + p_{12}\,z' + q_{11}\,y + q_{12}\,z &= 0, \\ z'' + p_{21}\,y' + p_{22}\,z' + q_{21}\,y + q_{22}\,z &= 0. \end{aligned} \tag{A}$$

In order to obtain system (A) from system (1) it is sufficient to differentiate the first two equations of system (1) and then eliminate ϱ, σ, ϱ', σ'. We prefer however to start from our canonical system (2). We find then that the coefficients of system (A), for our surface R_{yz}, are given by the following formulas in terms of the coefficients of system (2):

$$\begin{aligned} p_{11} &= (a_{21}\,a'_{12} - a_{22}\,a'_{11})/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{11} &= -(a_{11}\,b_{11} + a_{12}\,b_{21}), \\ p_{12} &= (a_{12}\,a'_{11} - a_{11}\,a'_{12})/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{12} &= -(a_{12}\,b_{12} + a_{12}\,b_{22}), \\ p_{21} &= (a_{21}\,a'_{22} - a_{22}\,a'_{21})/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{21} &= -(a_{21}\,b_{11} + a_{22}\,b_{21}), \\ p_{22} &= (a_{12}\,a'_{21} - a_{11}\,a'_{22})/(a_{11}\,a_{22} - a_{21}\,a_{12}), & q_{22} &= -(a_{21}\,b_{12} + a_{22}\,b_{22}). \end{aligned} \tag{3}$$

We are thus enabled to avail ourselves of the classical theory of a single ruled surface.

* Wilczynski, *Projective Differential Geometry*, Leipzig, 1906, p. 126.

3. GREEN-RECIPROCAL RULED SURFACES

G. M. Green's reciprocal relation R plays a significant part in projective differential geometry. This relation may be briefly formulated as follows.* Consider an arbitrary non-developable surface S. At an arbitrary point on this surface there is a tangent plane and an osculating quadric. A line l which lies in the tangent plane but does not pass through the point of contact, and a line l', which passes through the point of contact but does not lie in the tangent plane, are said to be reciprocal to each other in case they are reciprocal polars with respect to the osculating quadric. The totality of lines l forms a congruence, called a Γ congruence, and the reciprocal lines l' form the reciprocal Γ' congruence.

Let us draw a curve C on our fundamental surface S. Corresponding to every point of this curve we have a pair of reciprocal lines l and l'. All these lines l' form a ruled surface, which intersects S in C, and the reciprocal lines l also form a ruled surface. Such ruled surfaces we shall call Green-reciprocal ruled surfaces.

The generators of Green-reciprocal ruled surfaces are in a one-to-one correspondence and are skew to each other. It will therefore be possible to apply our general theory of pairs of ruled surfaces to Green-reciprocal ruled surfaces. For this purpose we shall need to formulate the relation R analytically.

Let the four homogeneous coördinates $y^{(1)}, \ldots, y^{(4)}$ of an arbitrary point P_y on a surface S_y be given as analytic functions of two independent variables u and v. Let us suppose that S_y is non-degenerate and non-developable, and is referred to its asymptotic net. Then the four functions y are a fundamental system of solutions of a completely integrable system of partial differential equations which may be reduced to the form†

$$y_{uu}+2by_v+fy=0, \qquad y_{vv}+2a'y_u+gy=0. \tag{4}$$

The points P_ρ and P_σ defined by

$$\rho = y_u-\beta y, \qquad \sigma = y_v-\alpha y, \tag{5}$$

where α and β are functions of u and v, lie in the tangent plane of S_y at P_y. More precisely, P_ρ lies on the line tangent at P_y to the asymptotic curve

* Green, Proceedings of the National Academy of Sciences, vol. 3 (1917) pp. 587–592.

† Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), p. 246.

$v = \text{const.}$ through P_y, and P_σ lies on the line tangent to the asymptotic $u = \text{const.}$ at P_y. The line $l_{\rho\sigma}$ which joins P_ρ and P_σ lies in the tangent plane of S_y at P_y, and does not pass through P_y. The line l_{yz} reciprocal to $l_{\rho\sigma}$ joins P_y to the point P_z defined by*

$$z = y_{uv} - \alpha y_u - \beta y_v. \tag{6}$$

An arbitrary curve C_y on S_y (except the curves $u = \text{const.}$) may be defined by expressing v as a function of u, in the form $v = v(u)$. As u varies, P_y moves along the curve C_y, and the lines l_{yz} and $l_{\rho\sigma}$ generate two Green-reciprocal ruled surfaces, R_{yz} and $R_{\rho\sigma}$.

We are now ready to set up the system (1) for Green-reciprocal ruled surfaces. We first calculate the partial derivatives of z, ϱ and σ with respect to u, and with respect to v, obtaining

$$\begin{aligned} z_u &= Py - \beta z + A\varrho - F'\sigma, \\ z_v &= Qy - \alpha z - G'\varrho + B\sigma, \end{aligned} \tag{7}$$

where we have placed

$$\begin{aligned} A &= 4a'b - \alpha_u - \alpha\beta, & F' &= f + \beta^2 + \beta_u - 2b\alpha + 2b_v, \\ B &= 4a'b - \beta_v - \alpha\beta, & G' &= g + \alpha^2 + \alpha_v - 2a'\beta + 2a_u, \\ P &= A\beta - F'\alpha + 2bg - f_v + f\alpha, \\ Q &= B\alpha - G'\beta + 2a'f - g_u + g\beta; \end{aligned} \tag{8}$$

and obtaining

$$\begin{aligned} \varrho_u &= -Fy - \beta\varrho - 2b\sigma, & \varrho_v &= (\alpha\beta - \beta_v)y + z + \alpha\varrho, \\ \sigma_v &= -Gy - 2a'\varrho - \alpha\sigma, & \sigma_u &= (\alpha\beta - \alpha_u)y + z + \beta\sigma, \end{aligned} \tag{9}$$

where we have placed

$$F = f + \beta^2 + \beta_u + 2b\alpha, \qquad G = g + \alpha^2 + \alpha_v + 2a'\beta. \tag{10}$$

* Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 87.

We next calculate the total derivatives of y, z, ϱ and σ with respect to u, using the formula $y' = y_u + v' y_v$, where v' is obtained by differentiating the equation $v = v(u)$ defining R_{yz} and $R_{\rho\sigma}$. We obtain in this way a system of equations of the form (1), whose coefficients have the following values:

$$c_{11} = \beta + v'\alpha, \quad c_{12} = 0, \quad a_{11} = 1, \quad a_{12} = v',$$

$$c_{21} = P + v'Q, \quad c_{22} = -\beta - v'\alpha, \quad a_{21} = A - v'G, \quad a_{22} = -F' + v'B,$$

(11)

$$d_{11} = -\beta + v'\alpha, \quad d_{12} = -2b, \quad b_{11} = -F + v'(\alpha\beta - \beta_v), \quad b_{12} = v',$$

$$d_{21} = -2v'a', \quad d_{22} = \beta - v'\alpha, \quad b_{21} = \alpha\beta - \alpha_u - v'G, \quad b_{22} = 1.$$

We shall make use of this system of equations whenever we wish to apply our general theory to Green-reciprocal ruled surfaces.

4. Intersector curves

A curve on R_{yz} will be called an *intersector curve* (with respect to $R_{\rho\sigma}$) in case the tangent at each point of the curve intersects the line $l_{\rho\sigma}$ which corresponds to the generator l_{yz} that passes through the point. A similar definition may be made for intersector curves on $R_{\rho\sigma}$.

We shall now determine the intersector curves on R_{yz}. Any point P_φ on a generator l_{yz} (except P_z) may be defined by setting $\varphi = y + \lambda z$, where λ is a function of x. As x varies, the locus of P_φ is a curve C_φ on R_{yz}. We wish to determine λ so that C_φ will be an intersector curve. The point φ', where $\varphi' = y' + \lambda z' + \lambda' z$, is a point on the tangent of C_φ. Substituting from the first two equations of system (1), we find

$$(12)\quad \varphi' = (c_{11} + \lambda c_{21})\, y + (\lambda' + c_{12} + \lambda c_{22})\, z + (a_{11} + \lambda a_{21})\, \varrho + (a_{12} + \lambda a_{22})\, \sigma.$$

Then an arbitrary point on the tangent of C_φ is given by an expression of the form $\varphi' + k\varphi$. If this tangent intersects $l_{\rho\sigma}$, there must exist a value of k so that the expression $\varphi' + k\varphi$ is a linear combination of ϱ and σ only. Therefore k and λ must satisfy the two conditions

$$c_{11} + \lambda c_{21} + k = 0, \qquad \lambda' + c_{12} + \lambda c_{22} + k\lambda = 0.$$

Eliminating k, we obtain the differential equation of the intersector curves on R_{yz},

$$\lambda' = -c_{12} + (c_{11} - c_{22})\lambda + c_{21}\lambda^2. \tag{13}$$

The corresponding equation for the intersector curves on $R_{\rho\sigma}$ may be deduced in the same way. If we denote an arbitrary point on $l_{\rho\sigma}$ by P_ψ, where $\psi = \varrho + \mu\sigma$, we find

$$\psi' = (b_{11} + \mu b_{21})y + (b_{12} + \mu b_{22})z + (d_{11} + \mu d_{21})\varrho + (\mu' + d_{12} + \mu d_{22})\sigma. \tag{14}$$

And the differential equation for the intersector curves on $R_{\rho\sigma}$ is

$$\mu' = -d_{12} + (d_{11} - d_{22})\mu + d_{21}\mu^2. \tag{15}$$

We observe that if our fundamental equations are written in the canonical form (2), then the curves $\lambda = \text{const.}$ are the intersector curves on R_{yz}, and the curves $\mu = \text{const.}$ are the intersector curves on $R_{\rho\sigma}$. In fact, inspection of system (1) will show that C_y is an intersector curve if $c_{12} = 0$, and C_z is an intersector curve if $c_{21} = 0$. Similarly, C_ρ is an intersector curve if $d_{12} = 0$, and C_σ is an intersector curve if $d_{21} = 0$.

We remark further that equation (13) is a Riccati equation. Therefore *there is a one-parameter family of intersector curves on R_{yz}, and they are determined by solving a Riccati equation.* Conversely, we can show that, if we have given a ruled surface R_{yz} and a Riccati equation

$$\lambda' = P + Q\lambda + R\lambda^2,$$

then there exists a ruled surface $R_{\rho\sigma}$ with respect to which the one-parameter family of curves defined by the given equation are intersector curves. Comparing the given equation with equation (13), let us choose $c_{11} = 0$, and set

$$c_{12} = -P, \qquad c_{22} = -Q, \qquad c_{21} = R.$$

If we introduce these values into the first two equations of system (1), and solve these two equations for ϱ and σ, we obtain

$$\varrho = Pz + y', \qquad \sigma = -Ry + (P + Q)z + y' + z'.$$

These formulas determine the required ruled surface $R_{\rho\sigma}$. Therefore we have shown that *the theory of a Riccati equation is equivalent to the theory of the intersector curves on a ruled surface with respect to some associated ruled surface.* It should be noted that our determination of $R_{\rho\sigma}$ is not unique, since only the difference $c_{11} - c_{22}$ is defined.

Since the cross ratio of any four particular solutions of a Riccati equation is constant, we see that, *if we select any four intersector curves on R_{yz}, they will cut each generator l_{yz} in a set of four points having the same cross ratio on all generators.*

The curved asymptotics on a ruled surface have long been known* to possess the cross ratio property which we have just established for intersector curves in general. We shall find a Riccati equation defining the curved asymptotics on R_{yz}, using our canonical system (2). Let P_φ, where $\varphi = y + \lambda z$, be an arbitrary point on l_{yz}. Then differentiating twice and making use of system (2), we find

$$\begin{aligned} \varphi' &= \lambda' z + (a_{11} + \lambda a_{21})\varrho + (a_{12} + \lambda a_{22})\sigma, \\ \varphi'' &= (\,)y + (\,)z + [(a_{11} + \lambda a_{21})' + \lambda' a_{21}]\varrho + [(a_{12} + \lambda a_{22})' + \lambda' a_{22}]\sigma, \end{aligned} \tag{16}$$

the omitted coefficients being immaterial. The locus C_φ of P_φ will be an asymptotic curve on R_{yz} if the osculating plane of C_φ at every point coincides with the tangent plane of R_{yz} at the point. Then the four points y, φ, φ', φ'' will be coplanar. The condition for their coplanarity reduces to

$$\begin{aligned} 2(a_{12}\,a_{21} - a_{11}\,a_{22})\,\lambda' = {} & a_{11}\,a'_{12} - a_{12}\,a'_{11} \\ & + (a_{11}\,a'_{22} - a_{22}\,a'_{11} + a_{21}\,a'_{12} - a_{12}\,a'_{21})\,\lambda + (a_{21}\,a'_{22} - a_{22}\,a'_{21})\,\lambda^2. \end{aligned} \tag{17}$$

This equation defines the curved asymptotics on R_{yz}.

A second property of the asymptotics may also be extended to intersector curves. Since an asymptotic tangent intersects three consecutive generators on a ruled surface, it follows that the asymptotic tangents of R_{yz} constructed at points of a fixed generator l_{yz} form a quadric. This quadric is called the osculating quadric of R_{yz} at l_{yz}. Now a tangent of an intersector curve on R_{yz} intersects two consecutive generators at l_{yz} and also intersects $l_{\rho\sigma}$. Therefore the intersector tangents of R_{yz} constructed at points of a fixed generator

* P. Serret, *Théorie Nouvelle Géométrique et Mécanique des Lignes à Double Courbure*, Paris, 1860.

l_{yz} also form a quadric, which we shall call the *intersector quadric* of R_{yz} at l_{yz}. In order to find the equation of this quadric, we observe that an arbitrary point on the tangent of an arbitrary intersector curve is given by $\varphi' + k\varphi$, where φ' has the value written in equation (12), and λ satisfies equation (13). If we introduce a local tetrahedron of reference with vertices at the points y, z, ϱ, σ, and with suitably chosen unit point, we may write the local coördinates of the point $\varphi' + k\varphi$ in the form

$$x_1 = c_{11} + \lambda c_{21} + k, \qquad x_3 = a_{11} + \lambda a_{21},$$

$$x_2 = (c_{11} + \lambda c_{21} + k)\lambda, \qquad x_4 = a_{12} + \lambda a_{22}.$$

Eliminating k and λ, and making the result homogeneous in the usual way, we obtain the equation of the intersector quadric of R_{yz} in the form

$$a_{12} x_1 x_3 - a_{11} x_1 x_4 + a_{22} x_2 x_3 - a_{21} x_2 x_4 = 0.$$

The relation of asymptotics to intersector curves is made clear by the following considerations. Suppose that $l_{\rho\sigma}$ happens to be a generator of the same set as l_{yz} on the osculating quadric of R_{yz}. Then $R_{\rho\sigma}$ is called a derivative ruled surface* of R_{yz}, and the intersector curves are the asymptotics. *The asymptotics on R_{yz} are intersector curves with respect to an arbitrary derivative ruled surface of R_{yz}.*

5. Analogues of flecnode curves

When four skew straight lines are given, there exist two other straight lines each of which intersects all four given lines. Indeed, the flecnodes on a generator of a ruled surface have been defined† as the two points at each of which a line may be drawn intersecting four consecutive generators. We wish to extend this notion of flecnode in two directions.

In the first place, we recall that an asymptotic tangent of R_{yz} intersects three consecutive generators l_{yz}. Therefore there are two points on each generator l_{yz} which are characterized by the fact that the asymptotic tangent at each of them intersects the corresponding line $l_{\rho\sigma}$. If $l_{\rho\sigma}$ happens to be also a generator of R_{yz}, and if we let $l_{\rho\sigma}$ approach l_{yz} over the surface R_{yz}, then our two points approach the flecnodes of l_{yz} as limiting positions.

* Wilczynski, *Projective Differential Geometry*, p. 147.

† Wilczynski, *Projective Differential Geometry*, p. 149.

We may determine our two points as follows. Let the fundamental equations be written in the canonical form (2), and let any point P_φ on l_{yz} be defined by $\varphi = y + \lambda z$. If the asymptotic tangent at P_φ coincides with the intersector tangent, then equation (17) of the asymptotics and the equation $\lambda' = 0$ of the intersector curves may be regarded as simultaneous. Therefore we determine the two points at which the asymptotic tangents are also intersector tangents by solving the quadratic

$$(18)\quad a_{11}\, a'_{12} + a_{12}\, a'_{11} + (a_{11}\, a'_{22} - a_{22}\, a'_{11} + a_{21}\, a'_{12} - a_{12}\, a'_{21})\,\lambda + (a_{21}\, a'_{22} - a_{22}\, a'_{21})\,\lambda^2 = 0.$$

The locus of each of these points is a curve analogous to a flecnode curve. At each of its points the asymptotic and intersector tangents coincide. This analogue of the flecnode curve is indeterminate in case the asymptotic and intersector curves coincide, that is, in case $R_{\rho\sigma}$ is a derivative of R_{yz}.

We arrive at our second analogue of a flecnode as follows. Since any tangent of R_{yz} intersects two consecutive generators l_{yz}, and since any tangent of $R_{\rho\sigma}$ intersects two consecutive generators $l_{\rho\sigma}$, it follows that there are two lines which are tangent to both R_{yz} and $R_{\rho\sigma}$, the points of contact of each line being on corresponding generators l_{yz} and $l_{\rho\sigma}$.

In order to determine the common tangent lines of R_{yz} and $R_{\rho\sigma}$, we first determine the tangent plane of each surface. The tangent plane of R_{yz} at a point P_φ, where $\varphi = y + \lambda z$, is determined by P_y, P_z and $P_{\varphi'}$, where φ' is given by equation (12). Therefore this plane is determined by l_{yz} and the point

$$(19)\qquad (a_{11} + \lambda a_{21})\,\varrho + (a_{12} + \lambda a_{22})\,\sigma,$$

where the plane intersects the generator $l_{\rho\sigma}$ corresponding to l_{yz}. Similarly, the tangent plane of $R_{\rho\sigma}$ at a point P_ψ, where $\psi = \varrho + \mu\sigma$, is determined by P_ρ, P_σ, and $P_{\psi'}$, where ψ' is given by equation (14). Therefore this plane is determined by $l_{\rho\sigma}$ and the point

$$(20)\qquad (b_{11} + \mu b_{21})\,y + (b_{12} + \mu b_{22})\,z,$$

where the plane intersects the generator l_{yz} corresponding to $l_{\rho\sigma}$.

Now the line of intersection of the tangent planes of R_{yz} and $R_{\rho\sigma}$ joins the points (19) and (20). This line is tangent to R_{yz} if the point (20) coincides

with P_φ, that is, if λ and μ satisfy the condition

$$\lambda = (b_{12} + \mu b_{22})/(b_{11} + \mu b_{21}). \tag{21}$$

Similarly, this line is tangent to $R_{\rho\sigma}$ if the point (19) coincides with P_ϕ, that is, if λ and μ satisfy the condition

$$\mu = (a_{12} + \lambda a_{22})/(a_{11} + \lambda a_{21}). \tag{22}$$

If we regard equations (21) and (22) as simultaneous and eliminate μ, we obtain

$$(a_{21}\, b_{11} + a_{22}\, b_{21})\, \lambda^2 + (a_{11}\, b_{11} + a_{12}\, b_{21} - a_{22}\, b_{22} - a_{21}\, b_{12})\, \lambda - (a_{11}\, b_{12} + a_{12}\, b_{22}) = 0. \tag{23}$$

Solution of this quadratic determines the two points on l_{yz} at which lines can be drawn tangent to both of R_{yz} and $R_{\rho\sigma}$. The points of contact of these common tangent lines on $R_{\rho\sigma}$ are determined by eliminating λ from equations (21) and (22) and solving the quadratic

$$(a_{11}\, b_{21} + a_{21}\, b_{22})\, \mu^2 + (a_{11}\, b_{11} + a_{21}\, b_{12} - a_{12}\, b_{21} - a_{22}\, b_{22})\, \mu - (a_{12}\, b_{11} + a_{22}\, b_{12}) = 0. \tag{24}$$

6. Applications to Green-reciprocal ruled surfaces

We shall now consider a pair of Green-reciprocal ruled surfaces and apply our general theory to them. For this purpose we employ equations (11). Since $c_{12} = 0$ in equations (11), it follows that C_y *is an intersector curve on* R_{yz}. Moreover, this fact is geometrically obvious, since the line tangent to C_y lies in the tangent plane of S_y and therefore intersects $l_{\mu\sigma}$. The equation (13) of the intersector curves on R_{yz} becomes

$$\lambda' = 2(\beta + v'\alpha)\, \lambda + (P + v'Q)\, \lambda^2. \tag{25}$$

Since a particular solution of this equation, namely $\lambda = 0$, is known, *the intersector curves on* R_{yz} *can be determined by quadratures.* In fact, if we

place $\lambda = 1/\nu$, equation (25) reduces to the linear equation

$$\nu' = -(P + v'Q) - 2(\beta + v'\alpha)\nu. \tag{26}$$

We might expect C_y to be also an asymptotic on R_{yz}. But if C_y is an asymptotic on R_{yz}, then the osculating plane of C_y is tangent to R_{yz} at P_y and therefore contains the generator l_{yz} through P_y. Then, using the language of Miss Sperry, we may say that *the curve C_y is an asymptotic on R_{yz} if, and only if, C_y is a union curve* of the congruence Γ'*. This theorem is a generalization of one of Green's theorems.† He considered an arbitrary conjugate net on S_y and the axis congruence of this net. Then his theorem asserts that the ruled surface of axes which intersects S_y in a curve of the fundamental conjugate net has this curve for an asymptotic. To derive Green's theorem from ours, we have only to note that a curve of the fundamental conjugate net is a union curve of the axis congruence of the net.

If we select any particular point on l_{yz}, the plane of this point and $l_{\rho\sigma}$ will touch $R_{\rho\sigma}$ in a definite point. Let us consider the point P_y. The plane of P_y and $l_{\rho\sigma}$, which is the tangent plane of S_y at P_y, touches $R_{\rho\sigma}$ in a point which is of particular interest. Let this point be P_ψ, where $\psi = \varrho + \mu\sigma$. Substituting from equations (11) into equations (20), we find that the tangent plane of $R_{\rho\sigma}$ at P_ψ intersects l_{yz} in the point

$$[-F + v'(\alpha\beta - \beta_v) + \mu(\alpha\beta - \alpha_u - v'G)]y + (v' + \mu)z. \tag{27}$$

This point coincides with P_y if $\mu = -v'$. Therefore the tangent plane of S_y at P_y, touches $R_{\rho\sigma}$ at the point $\varrho - v'\sigma$. But if we recall that the surface S_y is referred to its asymptotic net, we see that the point $\varrho - v'\sigma$ lies on the conjugate of the tangent to the curve C_y that corresponds to $R_{\rho\sigma}$. Therefore *the point of contact of the tangent plane of S_y with an arbitrary ruled surface $R_{\rho\sigma}$ lies on the conjugate of the tangent to the curve on S_y that corresponds to $R_{\rho\sigma}$.*

This theorem is a generalization of a theorem which has had an interesting history. It was first proved by Wilczynski for the directrix congruences, and by Sullivan for the scroll directrix congruences. Green proved it for an arbi-

* P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1918), pp. 213–224.

† G. M. Green, Second Memoir, American Journal of Mathematics, vol. 38 (1916), p. 303.

trary pair of reciprocal congruences. His theorem* amounts to this, that, for a ruled surface $R_{\rho\sigma}$ which is developable, the line $l_{\rho\sigma}$ touches the edge of regression of the developable in a point which lies on the conjugate of the tangent to the curve on S_y which corresponds to the developable. Our generalization is suggested by the fact that the point where $l_{\rho\sigma}$ touches the edge of regression of $R_{\rho\sigma}$, namely the focal point of $l_{\rho\sigma}$, is also the point where the tangent plane of S_y touches $R_{\rho\sigma}$. In fact, any plane containing $l_{\rho\sigma}$ will touch $R_{\rho\sigma}$ at the focal point of $l_{\rho\sigma}$.

When $R_{\rho\sigma}$ is developable the point (27) is indeterminate, and we have

$$v' + \mu = 0,$$

$$-F + v'(\alpha\beta - \beta_v) + \mu(\alpha\beta - \alpha_u - v'G) = 0. \tag{28}$$

If we eliminate μ we obtain Green's equation for determining the developables of the Γ congruence. And if we eliminate v', we obtain Green's equation for determining the focal sheets of the Γ congruence.†

The locus of the point P_ψ, where $\psi = \varrho - v'\sigma$, is a curve on $R_{\rho\sigma}$. We shall call this curve the *contact curve* on $R_{\rho\sigma}$, because at each of its points the tangent plane of S_y touches $R_{\rho\sigma}$. Let us enquire under what condition the contact curve C_ψ is also an intersector curve. Since C_ψ is an intersector curve its tangent at P_ψ must intersect l_{yz}, but, since C_ψ is the contact curve, its tangent must lie in the tangent plane of S_y at P_y. Therefore the tangent passes through P_y and is the conjugate of the tangent of the curve C_y which corresponds to $R_{\rho\sigma}$. The locus of all these conjugate tangents is a developable, called the conjugate developable of C_y, which, in our case, has C_ψ for its edge of regression. So at every point P_y the conjugate of the tangent of C_y touches the edge of regression of the conjugate developable of C_y at a point on $l_{\rho\sigma}$. Such a curve C_y has been called an adjoint union curve‡ of the Γ congruence. Therefore, *if the contact curve on $R_{\rho\sigma}$ is also an intersector curve, then the curve C_y which corresponds to $R_{\rho\sigma}$ is an adjoint union curve of the Γ congruence.* If we wish to prove this theorem analytically, we may substitute from equation (11) into equation (15) and replace therein μ by $-v'$. The result is the well known differential equation of the second order for the adjoint union curves on S_y.

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* Green, *Memoir on the general theory of surfaces*, these Transactions, vol. 20 (1919), p. 94.

† Green, loc. cit., p. 89 and p. 91.

‡ Green, loc. cit., p. 140.

Let us again employ the geometrical argument to demonstrate still another theorem concerning the contact curve on $R_{\rho\sigma}$. The tangent planes of S_y, constructed at points of the curve C_y, are also the tangent planes of $R_{\rho\sigma}$ constructed at points of the contact curve. These planes envelope the conjugate developable of C_y, and, if the contact curve is also an intersector curve, the conjugate developable is the developable of the tangents of the contact curve. Since the planes enveloping a developable are also the osculating planes of the edge of regression of the developable, it follows in our case that at each point of the contact curve the osculating plane of the curve is the tangent plane of $R_{\rho\sigma}$. Therefore, *if the contact curve is an intersector curve, then it is an asymptotic.* This theorem may also be proved analytically by direct calculation.

7. The directrix congruences

One of the most important examples of pairs of congruences reciprocal with respect to a surface is the pair of directrix congruences* of the surface. Wilczynski defined the directrix congruences in terms of the osculating linear complexes of the asymptotic curves on the surface. We are able to give another geometrical characterization of them. Let us substitute from equations (11) into equation (24). We obtain in this way the equation

$$\begin{aligned}(29)\qquad &[4a'b - 2\alpha_u - v'(G+G')]\mu^2 + [F' - F - v'^2(G'-G)]\mu \\ &+ v'[F + F' - v'(4a'b - 2\beta_v)] = 0,\end{aligned}$$

which determines the two points $l_{\rho\sigma}$ at which tangents to $R_{\rho\sigma}$ can be drawn which also touch R_{yz} at points on the corresponding generator l_{yz}. Now these two points separate P_ρ and P_σ harmonically if, and only if,

$$F' - F - v'^2(G'-G) = 0.$$

This equation is satisfied for every v', that is, for every pair of Green-reciprocal ruled surfaces, if, and only if,

$$F' = F, \qquad G' = G.$$

* Wilczynski, *Curved surfaces*, Second Memoir, these Transactions, vol. 9 (1908), pp. 79–120.

Reference to equations (8) and (10) will show that these equations are equivalent to

$$\alpha = b_v/2b, \qquad \beta = a'_u/2a'.$$

Therefore the congruences Γ and Γ' are the directrix congruences.* Consequently *we may characterize the directrix congruences by saying that they are the only reciprocal congruences which have the property that, for every pair of Green-reciprocal ruled surfaces R_{yz} and $R_{\rho\sigma}$, the two points, on $l_{\rho\sigma}$, at which tangents of $R_{\rho\sigma}$ can be drawn which are also tangent to R_{yz} at points on l_{yz}, separate P_ρ and P_σ harmonically.*

* G. M. Green, *Memoir on the general theory of surfaces,* these Transactions, vol. 20 (1919), p. 92.

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SYMMETRIC TENSORS OF THE SECOND ORDER WHOSE FIRST COVARIANT DERIVATIVES ARE ZERO*

BY

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1. Consider a Riemann space of the nth order, whose fundamental quadratic form, assumed to be positive definite, is written

$$ds^2 = g_{rs}\,dx^r\,dx^s \qquad (g_{rs} = g_{sr}), \tag{1}$$

where r and s are summed from 1 to n in accordance with the usual convention which will be followed throughout this paper. It is well known that the first covariant derivatives $g_{rs|t}$ are zero, where

$$g_{rs\,t} = \frac{\partial g_{rs}}{\partial x^t} - g_{ra}\,\Gamma^a_{st} - g_{as}\,\Gamma^a_{rt} \tag{2}$$

and

$$\Gamma^a_{st} = \tfrac{1}{2}\,g^{ap}\left(\frac{\partial g_{sp}}{\partial x^t} + \frac{\partial g_{tp}}{\partial x^s} - \frac{\partial g_{st}}{\partial x^p}\right), \tag{3}$$

the function g^{ap} being the cofactor of g_{ap} in the determinant

$$g = |g_{rs}| \tag{4}$$

divided by g. It is the purpose of this paper to determine the necessary and sufficient conditions that there exist a symmetric covariant tensor α_{rs} such that the first covariant derivatives $\alpha_{rs|t}$ are zero, or more than one such tensor.

2. Let α_{rs} denote the covariant components of any symmetric tensor of the second order. If ϱ_h is a root of the equation

$$|\alpha_{rs} - \varrho\, g_{rs}| = 0, \tag{5}$$

the functions λ^r_h $(r = 1, \ldots, n)$ defined by

$$(\alpha_{rs} - \varrho_h\, g_{rs})\,\lambda^r_h = 0 \qquad (s = 1, \ldots, n) \tag{6}$$

* Presented to the Society, April 28, 1923.

are the contravariant components of a vector. It is well known that the roots of (5) are real, and that if they are simple, the n corresponding vectors at a point are mutually orthogonal.* Moreover, if a root is of order m, equations (6) admit m sets of independent solutions, and any linear combination of them is also a solution. It is possible to choose m solutions so that the corresponding vectors at a point are mutually orthogonal, and thus from (6) obtain n sets of solutions so that the corresponding vectors at a point are orthogonal; that is,

$$g_{rs}\lambda_h^r\lambda_k^s = 0 \qquad (h,k = 1,\ldots,n;\ h \neq k). \tag{7}$$

Moreover, the components may be chosen so that

$$g_{rs}\lambda_h^r\lambda_h^s = 1 \qquad (h = 1,\ldots,n), \tag{8}$$

that is, the vectors are unit vectors.

The curves in space whose direction at each point is defined by λ_h^r form a congruence of curves C_h. Thus equations (6) define an n-uple of congruences of curves, such that the curves of the n-uple through a point are mutually orthogonal.

The covariant components $\lambda_{h,r}$ of the vector h are given by

$$\lambda_{h,r} = g_{rs}\lambda_h^s, \qquad \lambda_h^s = g^{rs}\lambda_{h,r}, \tag{9}$$

and hence (7) and (8) are equivalent to

$$\lambda_{h,r}\lambda_k^r = \delta_{hk}, \tag{10}$$

where

$$\delta_{hk} = 1 \text{ for } h = k; \quad = 0 \text{ for } h \neq k. \tag{11}$$

The functions γ_{hij} defined by

$$\gamma_{hij} = \lambda_{h,r/s}\lambda_i^r\lambda_j^s, \tag{12}$$

where $\lambda_{h,r/s}$ is the covariant derivative of $\lambda_{h,r}$ with respect to x^s, are invariants; they are called *rotations* by Ricci and Levi-Civita.† They have shown that

$$\gamma_{hij} + \gamma_{hji} = 0, \qquad \gamma_{hhi} = 0 \qquad (h,i,j = 1,\ldots,n). \tag{13}$$

* Cf. these Transactions, vol. 25 (1923), p. 259.

† Mathematische Annalen, vol. 54 (1901), p. 148; also, Wright, *Invariants of Quadratic Differential Forms*, Cambridge Tract, No. 9, p. 68.

From (12) we have

$$\lambda_{h,r/s} = \sum_{i,j}^{1\ldots n} \gamma_{hij}\, \lambda_{i,r}\, \lambda_{j,s}, \tag{14}$$

and since $g_{rs/t} = 0$, it follows from (9) that

$$\lambda^p_{h/s} = \sum_{i,j}^{1\ldots n} \gamma_{hij}\, \lambda^p_i\, \lambda_{j,s}. \tag{15}$$

3. If all the roots of (5) are equal, we must have $\alpha_{rs} = \varrho\, g_{rs}$. Differentiating covariantly with respect to x^t, and making use of the fact that $g_{rs/t} = 0$ and the assumption that $\alpha_{rs/t} = 0$, we have that ϱ is constant. Consequently α_{rs} is essentially the same as g_{rs}. We exclude this case from further consideration.

Since (7) is satisfied whether the functions λ^r_h and λ^r_k correspond to different simple roots of (5), or to the same multiple root when such exists, we have from (6)

$$\alpha_{rs}\, \lambda^r_h\, \lambda^s_k = 0 \qquad (h, k = 1, \ldots, n;\ h \neq k). \tag{16}$$

Also from (6) we have

$$\alpha_{rs}\, \lambda^r_h\, \lambda^s_h = \varrho_h. \tag{17}$$

From (17) we have by differentiating covariantly with respect to x^t and making use of (15), (16), and (17)

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_h = \frac{\partial \varrho_h}{\partial x^t}. \tag{18}$$

Also from (16) we have, because of (13), (14), (16) and (17),

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_k + \sum_{j}^{1\ldots n} (\varrho_k - \varrho_h)\, \gamma_{hkj}\, \lambda_{j,t} = 0.$$

Multiplying by λ^t_l and summing for t, we have

$$\alpha_{rs/t}\, \lambda^r_h\, \lambda^s_k\, \lambda^t_l + (\varrho_k - \varrho_h)\, \gamma_{hkl} = 0 \qquad (h \neq k). \tag{19}$$

From (18) it follows that if $\alpha_{rs/t} = 0$ the roots ϱ are constant. And from (19) we have for two different roots

$$\gamma_{hkl} = 0 \qquad (h \neq k). \tag{20}$$

Let ϱ_1 be a root of (5) which we assume to be a multiple root of order m, and denote by λ_h^r $(h = 1, \ldots, m)$ the components of the m mutually orthogonal vectors corresponding to it, and by λ_k^r $(k = m+1, \ldots, n)$ the components of the directions corresponding to the other roots of (5). From (20) we have

$$\gamma_{hkl} = 0 \qquad (h = 1, \ldots, m;\ k = m+1, \ldots, n;\ l = 1, \ldots, n). \tag{21}$$

Consider the system of equations

$$X_k(f) \equiv \lambda_k^r \frac{\partial f}{\partial x^r} = 0 \qquad (k = m+1, \ldots, n). \tag{22}$$

If we introduce the notation

$$\frac{\partial f}{\partial s^k} = \lambda_k^r \frac{\partial f}{\partial x^r},$$

then, as Ricci and Levi-Civita have shown*, the relation

$$\frac{\partial}{\partial s_j}\frac{\partial f}{\partial s_k} - \frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_j} = \sum_i^{1\ldots n}(\gamma_{ijk} - \gamma_{ikj})\frac{\partial f}{\partial s_i} \tag{23}$$

is satisfied for any function f.

Applying this formula to equations (22) we have in consequence of (21)

$$X_j X_k(f) - X_k X_j(f) = \sum_i^{m+1\ldots n}(\gamma_{ijk} - \gamma_{ikj}) X_i(f) \qquad (j, k = m+1, \ldots, n).$$

Hence the system (22) is complete and admits m independent solutions, say f_h $(h = 1, \ldots, m)$.

* Loc. cit., p. 150; Wright, p. 69.

Let ϱ_2 be another root of (5), of order p, and denote by λ_j^r $(j = m+1, \ldots, m+p)$ the components of the corresponding vectors. In like manner we show that the equations

$$\lambda_l^r \frac{\partial f}{\partial x^r} = 0 \qquad (l = 1, \ldots, m, m+p+1, \ldots, n)$$

form a complete system and admit p independent solutions f_j $(j = m+1, \ldots, m+p)$.

From (22) and the equations

$$\lambda_k^r \lambda_{h,r} = 0 \qquad (h = 1, \ldots, m; k = m+1, \ldots, n)$$

it follows that there exist functions a_h^σ such that

$$\frac{\partial f_h}{\partial x^r} = \sum_\sigma a_h^\sigma \lambda_{\sigma,r} \qquad (h, \sigma = 1, \ldots, m).$$

In like manner, we have

$$\frac{\partial f_j}{\partial x^r} = \sum_\tau b_j^\tau \lambda_{\tau,r} \qquad (j, \tau = m+1, \ldots, m+p).$$

Consequently we have

$$g^{rs} \frac{\partial f_h}{\partial x^r} \frac{\partial f_j}{\partial x^s} = \sum_{\sigma,\tau} a_h^\sigma b_j^\tau g^{rs} \lambda_{\sigma,r} \lambda_{\tau,s} = 0,$$

that is, any hypersurface $f_h = \text{const.}$ is orthogonal to each of the hypersurfaces $f_j = \text{const.}$

Proceeding in this manner with the other roots of (5) we obtain a group of hypersurfaces corresponding to each distinct root of (5), the number of hypersurfaces in a group being equal to the order of the root. Any two hypersurfaces of two different groups are orthogonal to one another. If we take these n families of hypersurfaces for the parametric surfaces $x^r = \text{const.}$ $(r = 1, \ldots, n)$, it follows that the functions g_{rs} are zero, for the case where $x^r = \text{const.}$ and $x^s = \text{const.}$ are hypersurfaces of different groups; in this sense we say that r and s refer to different groups, or different roots of (5).

From the equations (22) for this choice of the variables x, it follows that $\lambda_k^r = 0$, for r and k referring to different roots of (5). From (9) it follows also that $\lambda_{k,r} = 0$ for k and r referring to different roots.

Equations (6) may be replaced by*

$$\alpha_{rs} = \sum_{h}^{1\ldots n} \varrho_h \lambda_{h,r} \lambda_{h,s} \tag{24}$$

whether the roots of (5) are simple, or some are multiple. From (24) and the preceding observations it follows

$$\begin{aligned} \alpha_{rs'} &= g_{rs'} = 0, \\ \alpha_{rs} &= \varrho_h g_{rs}, \end{aligned} \tag{25}$$

where r and s' refer to any two different roots and r and s refer to the root ϱ_h.†

4. From (25) we have $\alpha_{rs'} = 0$, hence if $\alpha_{rs'/t} = 0$, we must have (cf. (2))

$$\alpha_{rl}\Gamma^l_{s't} + \alpha_{s'q}\Gamma^q_{rt} = 0 \qquad (l, q = 1, \ldots, n),$$

that is

$$\alpha_{rl}\, g^{l\beta}\left[\frac{\partial g_{s'\beta}}{\partial x^t} + \frac{\partial g_{t\beta}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^\beta}\right] + \alpha_{s'q}\, g^{q\beta}\left[\frac{\partial g_{r\beta}}{\partial x^t} + \frac{\partial g_{t\beta}}{\partial x^r} - \frac{\partial g_{rt}}{\partial x^\beta}\right] = 0.$$

If r refers to the root ϱ_1 of (5), say $r = 1, \ldots, m$ and s' to the root ϱ_2, say $s' = m+1, \ldots, m+p$, we have from (25)

$$\begin{aligned} \alpha_{rl} &= \varrho_1 g_{rl} && (l = 1, \ldots, m); \\ \alpha_{rl} &= 0 && (l = m+1, \ldots, n); \\ \alpha_{s'q} &= \varrho_2 g_{s'q} && (q = m+1, \ldots, m+p); \\ \alpha_{s'q} &= 0 && (q = 1, \ldots, m, m+p+1, \ldots, n). \end{aligned}$$

Hence the above equation reduces to

$$\varrho_1\left(\frac{\partial g_{s'r}}{\partial x^t} + \frac{\partial g_{tr}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^r}\right) + \varrho_2\left(\frac{\partial g_{rs'}}{\partial x^t} + \frac{\partial g_{ts'}}{\partial x^r} - \frac{\partial g_{rt}}{\partial x^{s'}}\right) = 0.$$

* Cf. Ricci and Levi-Civita, loc. cit., p. 159.

† Cf. Levi-Civita, Annali di Matematica, ser. 2, vol. 24 (1896), p. 298.

If now t and r refer to the same root, this equation reduces to

$$(\varrho_1 - \varrho_2)\frac{\partial g_{tr}}{\partial x^s} = 0,$$

and if t and s' refer to the same root, we have

$$(\varrho_1 - \varrho_2)\frac{\partial g_{s't}}{\partial x^r} = 0.$$

If r, s' and t refer to three different roots, the equation vanishes identically.

Since ϱ_1 and ϱ_2 are not equal by hypothesis, we have that each function g_{rs} depends only on the coördinates referring to the same root as r and s.

Consider again

$$a_{rs} = \varrho_1 g_{rs} \qquad (r, s = 1, \ldots, m).$$

Now

$$a_{rs|t} = \varrho_1 \frac{\partial g_{rs}}{\partial x^t} - a_{rl}\Gamma^l_{st} - a_{sl}\Gamma^l_{rt} \qquad (l = 1, \ldots, n),$$

which by (25) is reducible to

$$a_{rs|t} = \varrho_1\left(\frac{\partial g_{rs}}{\partial x^t} - g_{rl}\Gamma^l_{st} - g_{sl}\Gamma^l_{rt}\right) = \varrho_1 g_{rs|t} = 0.$$

Hence we have the following theorem:

A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order a_{rs} other than, with a positive definite fundamental form (1), g_{rs}, *such that its first covariant derivative is zero, is that* (1) *be reducible to a sum of forms*

$$\varphi^{(i)} = g^{(i)}_{r_i s_i}\, dx^{r_i}\, dx^{s_i}, \tag{26}$$

where $g^{(i)}_{r_i s_i}$ are functions at most of the x's of that form; then

$$a_{rs}\, dx^r\, dx^s = \sum_i \varrho_i \varphi^{(i)}, \tag{27}$$

where the ϱ's are arbitrary constants.

In particular, if all the roots of (5) are simple, the space is euclidean; if its fundamental form is taken in the form

$$ds^2 = \sum_i dx^{i^2} \qquad (i = 1, \ldots, n),$$

then

$$a_{rs}\, dx^r\, dx^s = \sum_i \varrho_i\, dx^{i^2} \qquad (i = 1, \ldots, n),$$

where the ϱ's are n different arbitrary constants.

When any one of the roots of (5) is simple, the corresponding congruence is normal, and the tangents to the congruence form a field of parallel vectors in the sense of Levi-Civita.*

5. In this section it will be shown that the problem of determining whether a given Riemann space admits one, or more, symmetric tensors whose first covariant derivatives are zero is a problem of algebra.†

We recall that if a_{rs} is any symmetric tensor, then

$$a_{rs|jk} - a_{rs|kj} = a_{rt}\, B^t_{sjk} + a_{st}\, B^t_{rjk},$$

where $a_{rs|jk}$ is the second covariant derivative of a_{rs}, and B^t_{sjk} are the components of the Riemann tensor of the second kind formed with respect to (1). If then $a_{rs|j} = 0$, we must have

$$a_{rs|jk} - a_{rs|kj} = 0, \tag{28}$$

and consequently we have equations of the form

$$a_{rt}\, B^t_{sjk} + a_{st}\, B^t_{rjk} = 0 \qquad (j, k, r, s, t = 1, \ldots, n). \tag{29}$$

Differentiating these equations covariantly successively we have the sets of equations

$$\begin{aligned} &a_{rt}\, B^t_{sjk|m_1} + a_{st}\, B^t_{rjk|m_1} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{rt}\, B^t_{sjk|m_1 \ldots m_l} + a_{st}\, B^t_{rjk|m_1 \ldots m_l} = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{30}$$

* Cf. Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 211.

† Cf. Eisenhart and Veblen, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 23; also Veblen and Thomas, these Transactions, vol. 25 (1923).

Since g_{rs} satisfies (28) the systems (29) and (30) are satisfied by g_{rs} and consequently are algebraically consistent. From this it follows either that the functions g_{rs} are the only solution of (29) and (30), or that (29) and the first $l\,(\geqq 0)$ sets of (30) admit a complete system of solutions g_{rs} and $\alpha_{rs}^{(1)}, \ldots, \alpha_{rs}^{(p)}$ which satisfy also the $(l+1)$th set of equations (30). In the latter case the general solution is of the form

$$\alpha_{rs} = \varphi^{(0)} g_{rs} + \varphi^{(1)} \alpha_{rs}^{(1)} + \cdots + \varphi^{(p)} \alpha_{rs}^{(p)}. \tag{31}$$

If any one of the functions $\alpha_{rs}^{(\sigma)}\,(\sigma = 1, \ldots, p)$ is substituted in (29) and the first l sets of (30), and these equations are differentiated covariantly, we have, in consequence of the above requirement, that the functions $\alpha_{rs|m}^{(\sigma)}$ $(\sigma = 1, \ldots, p;\ m = 1, \ldots, n)$ satisfy (29) and the first l sets of (30). Consequently we have

$$\alpha_{rs|m}^{(\sigma)} = \lambda_m^{(\sigma 0)} g_{rs} + \lambda_m^{(\sigma 1)} \alpha_{rs}^{(1)} + \cdots + \lambda_m^{(\sigma p)} \alpha_{rs}^{(p)}, \tag{32}$$

where the $p(p+1)$ vectors $\lambda_m^{(\sigma\beta)}\,(\sigma = 1, \ldots, p;\ \beta = 0, 1, \ldots, p)$ must be such that the functions (32) shall satisfy (28). Substituting in these equations we find that the functions λ must satisfy the system

$$\frac{\partial \lambda_p^{(\sigma\tau)}}{\partial x^q} - \frac{\partial \lambda_q^{(\sigma\tau)}}{\partial x^p} + \sum_{\omega} \left(\lambda_p^{(\sigma\omega)} \lambda_q^{(\omega\tau)} - \lambda_q^{(\sigma\omega)} \lambda_p^{(\omega\tau)}\right) = 0 \quad \begin{pmatrix} \sigma, \omega = 1, \ldots, p \\ \tau = 0, 1, \ldots, p \end{pmatrix}. \tag{33}$$

In order that α_{rs} given by (31) shall satisfy $\alpha_{rs|t} = 0$, it is necessary and sufficient that the functions $\varphi^{(i)}$ satisfy

$$\frac{\partial \varphi^{(0)}}{\partial x^t} + \sum_{\sigma} \varphi^{(\sigma)} \lambda_t^{(\sigma 0)} = 0 \qquad (\sigma = 1, \ldots, p), \tag{34}$$

and

$$\frac{\partial \varphi^{(\tau)}}{\partial x^t} + \sum_{\sigma} \varphi^{(\sigma)} \lambda_t^{(\sigma\tau)} = 0 \qquad (\sigma, \tau = 1, \ldots, p). \tag{35}$$

In consequence of (33) equations (35) are completely integrable and therefore admit solutions involving p arbitrary constants. Because of (33) the conditions of integrability of (34) are satisfied; hence $\varphi^{(0)}$ involves these p arbitrary constants and an additive arbitrary constant which may be neglected.*

* If a_{rs} is a tensor whose first covariant derivative is zero, so also is $a_{rs} + \lambda g_{rs}$, where λ is an arbitrary constant.

In view of the above results we have the theorem:

If equations (29) *and the first* $l(\geqq 0)$ *sets of equations* (30) *admit a complete system of solutions* g_{rs} *and* $\alpha_{rs}^{(\sigma)}$ $(\sigma = 1, \ldots, p)$ *which are also solutions of the* $(l+1)$*th set of equations* (30), *there exists a symmetric tensor of the second order, involving* p *arbitrary constants, whose first covariant derivative is zero.*

6. Suppose that the fundamental form is the sum of j forms (26). By definition

$$B^a_{prs} = g^{aq} B_{pqrs}, \tag{36}$$

where B_{pqrs} is the covariant Riemann tensor of the fourth order, that is,

$$B_{pqrs} = \frac{1}{2}\left(\frac{\partial^2 g_{ps}}{\partial x^q \partial x^r} + \frac{\partial^2 g_{qr}}{\partial x^p \partial x^s} - \frac{\partial^2 g_{pr}}{\partial x^q \partial x^s} - \frac{\partial^2 g_{qs}}{\partial x^p \partial x^r}\right) + g^{lm}(\Gamma_{ps,m}\Gamma_{qr,l} - \Gamma_{pr,m}\Gamma_{qs,l}), \tag{37}$$

where

$$\Gamma_{ps,m} = \frac{1}{2}\left(\frac{\partial g_{pm}}{\partial x^s} + \frac{\partial g_{sm}}{\partial x^p} - \frac{\partial g_{ps}}{\partial x^m}\right). \tag{38}$$

For the case under consideration, namely (26), it is readily shown that the components B_{pqrs} are zero, unless p, q, r, s refer to the same root of (5); likewise B^a_{prs}, and its first covariant derivatives $B^a_{prs|t}$. Consequently equations (29) and the first set of (30) admit, in addition to g_{rs}, the j sets of solutions of the form (25). If it is understood that each of the forms (26) is not further reducible to sums of such forms, we have a complete set of solutions of (29). Hence when the space is referred to the coördinates giving (25) the number l in the preceding theorem is zero.

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